Unit C1
Linear equations and matrices

Introduction to Book C

Systems of linear equations in several variables arise in areas as diverse as science, technology and economics. The solutions to such systems can provide answers to a wide range of problems, from supply and demand dependencies in economics to working out currents in electrical networks. Apart from such practical applications, solving systems of linear equations is also an interesting mathematical problem in itself. Much effort has been devoted to solving systems of linear equations. You will see that this process is not always straightforward, especially if the number of variables is large.

One key issue is whether a given system of linear equations has any solutions at all. As a specific example of the types of situation that may occur, consider the following three rather similar pairs of linear equations in the variables x and y.

$$\begin{aligned} x + 3y &= 5 \\ -2x + 6y &= 2 \end{aligned} \qquad \begin{aligned} -x + 3y &= 5 \\ -2x + 6y &= 2 \end{aligned} \qquad \begin{aligned} -x + 3y &= 1 \\ -2x + 6y &= 2 \end{aligned}$$

The first pair of equations has the unique solution $x = 2$, $y = 1$, whereas the second pair has no solutions, and the third pair has infinitely many solutions (for example, $x = -1$, $y = 0$, and $x = 0$, $y = \frac{1}{3}$). You will see that these different outcomes may be understood:

- algebraically, by studying the *matrix* of coefficients of the equations, and introducing a function of these coefficients, called the *determinant*
- geometrically, by interpreting solutions of the equations as points of intersection of the corresponding pairs of straight lines drawn in an (x, y)-plane.

The algebraic approach as well as the geometric approach can be generalised for systems of linear equations that involve more than two variables. The geometric approach will require us to use a generalisation of the plane called *n-dimensional Euclidean space*, whose elements are of the form $(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are real numbers. Depending on the context, we will interpret these elements as either *points* or *vectors* in Euclidean space. Although it is only easy to visualise objects in n-dimensional space when $n = 1$, 2 or 3, this more general Euclidean space is a convenient environment in which to develop the theory needed to analyse the solutions of systems of linear equations.

You will see that a key tool in this theory is the concept of a *linear transformation* which, in its basic form, is a function from one Euclidean space to another that preserves certain aspects of the geometric structure of the Euclidean space. For example, the function

$$t(x, y) = (x + 3y, -2x + 6y)$$

is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, which is closely related to the first pair of equations above. Indeed, solving that pair of equations is equivalent to finding a point (x, y) in $\mathbb{R}^2$ such that the function t maps

(x, y) to the point $(5, 2)$. This suggests that we can obtain information about the solutions of systems of linear equations by studying the corresponding linear transformations.

But linear transformations arise in situations apart from that of solving equations. For example, they are needed to manipulate computer graphic images, as illustrated in Figure 1.

Figure 1 The effect of a rotation, reflection and shear on an image

Finally, the range of available linear transformations can be increased greatly by introducing the notion of a *vector space*. This is a generalisation of n-dimensional Euclidean space, and it may be finite-dimensional or infinite-dimensional. The elements of a vector space are sometimes called *vectors*, but they can be very general objects; for example, you will look at vector spaces whose elements are real functions, and linear transformations between such vector spaces that arise from operations on real functions such as differentiation and integration. In this way, vector spaces and their associated linear transformations form a very general context in which many seemingly unrelated problems can be studied using similar techniques.

In this book on linear algebra you will learn about all these concepts: solving systems of linear equations, matrices, vector spaces and linear transformations. You will also use this theory to classify conics and quadrics.

Introduction

In this first unit of linear algebra you will begin by considering systems of linear equations in two and three unknowns. You will then see how matrices can be used as a concise way of representing systems of linear equations, before going on to study matrices themselves. You will see how properties of the matrix of coefficients may be used to quickly determine whether the system of linear equations has a unique solution.

Many of the ideas and methods you will meet in this unit will also be used in the subsequent three units on linear algebra.

1 Systems of linear equations

In this section you will revise systems of linear equations in two and three unknowns and see how these ideas extend to systems in more unknowns.

Recall that a **system of linear equations** in two (or three) unknowns is a collection of linear equations each written in terms of a set of two (or three) unknowns. A **solution** to a system of linear equations is an assignment of values to the unknowns that makes all the equations hold simultaneously; therefore such a system is also called a **system of simultaneous linear equations** in the given set of unknowns.

1.1 Systems in two and three unknowns

Systems in two unknowns: one equation

In Unit A1 *Sets, functions and vectors*, you saw that an equation of the form

$$ax + by = c$$

where a, b and c are real numbers, and a and b are not both zero, represents a line in $\mathbb{R}^2$. There are infinitely many solutions to this equation – one corresponding to each point on the line.

Systems in two unknowns: two equations

The solutions to the following system of two linear equations

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

in the two unknowns x and y, where $a, b, \ldots, f$ are real numbers, correspond to the points of intersection of these two lines in $\mathbb{R}^2$.

Now, two arbitrary lines in $\mathbb{R}^2$ may intersect at a unique point, be parallel, or coincide, which means that solving a system of two linear equations in two unknowns yields exactly one of the following three situations.

- There is a unique solution, when the two lines represented by the equations intersect at a unique point, as illustrated in Figure 2.

  For example, the system

  $$\begin{aligned} x - y &= -1 \\ 2x + y &= 4 \end{aligned}$$

  has the unique solution $x = 1$, $y = 2$, corresponding to the unique point of intersection $(1, 2)$ of the two lines in $\mathbb{R}^2$.

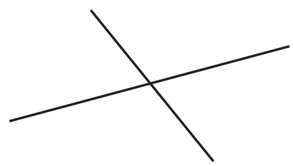

Figure 2 Two lines intersecting at a unique point

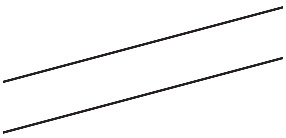

Figure 3 Two parallel lines with no point of intersection

- There is no solution, when the two lines represented by the equations are parallel, as illustrated in Figure 3.

  For example, the system

  $$\begin{aligned} x - y &= -1 \\ x - y &= 1 \end{aligned}$$

  represents two parallel lines in $\mathbb{R}^2$ that do not intersect, and so the system has no solution.

- There are infinitely many solutions, when the two lines represented by the equations coincide, as illustrated in Figure 4.

  For example, the system

  $$\begin{aligned} -6x + 3y &= -6 \\ 2x - y &= 2 \end{aligned}$$

  has infinitely many solutions, as the two equations represent the same line in $\mathbb{R}^2$: the equations are a multiple of one another. In a sense, the two lines intersect at all of their points; that is, each pair of values for x and y satisfying $2x - y = 2$ is a solution to this system.

Figure 4 Two coincident lines

Systems in three unknowns: one equation

In Unit A1 you saw that an equation of the form

$$ax + by + cz = d$$

where a, b, c and d are real numbers, and a, b and c are not all zero, represents a plane in $\mathbb{R}^3$. There are infinitely many solutions to this equation – one corresponding to each point in the plane.

Systems in three unknowns: two equations

The solutions to the system of two linear equations

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \end{aligned}$$

in the three unknowns x, y and z, where $a, b, \ldots, h$ are real numbers, correspond to the points of intersection of these two planes in $\mathbb{R}^3$.

Two arbitrary planes in $\mathbb{R}^3$ may intersect, be parallel or coincide. In general, when two distinct planes in $\mathbb{R}^3$ intersect, the set of common points is a line that lies in both planes. This means that solving a system of two linear equations in three unknowns yields exactly one of the following two situations.

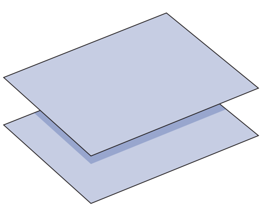

Figure 5 Two parallel planes

- There is no solution, when the two planes represented by the equations are parallel, as illustrated in Figure 5.

  For example, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

  represents two parallel planes in $\mathbb{R}^3$ and so has no solutions.

- There are infinitely many solutions, when the two planes represented by the equations coincide, or when they intersect in a line, as illustrated in Figures 6 and 7, respectively.

  For example,

  $$\begin{aligned} x + y + z &= 1 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

  has infinitely many solutions, as the two equations represent the same plane in $\mathbb{R}^3$. Each set of values for x, y and z satisfying $x + y + z = 1$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = -2$, $y = 4$, $z = -1$.

  Similarly, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \end{aligned}$$

  has infinitely many solutions: the planes in $\mathbb{R}^3$ represented by the two equations intersect in a line. The z-coordinate of each point on this line is zero, and so the line lies in the (x, y)-plane. Each set of values for x, y and z satisfying $x + y = 1$ and $z = 0$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = 5$, $y = -4$, $z = 0$.

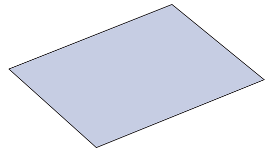

Figure 6 Two coincident planes

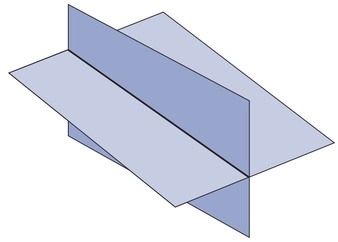

Figure 7 Two planes intersecting in a line

Systems in three unknowns: three equations

In a similar way, the solutions to the system of three linear equations

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned}$$

in the three unknowns x, y and z, where $a, b, \ldots, l$ are real numbers, correspond to the points of intersection of these three planes in $\mathbb{R}^3$.

Three arbitrary planes in $\mathbb{R}^3$ may meet each other in a number of different ways. We illustrate these possibilities below. A system of three linear equations in three unknowns yields exactly one of the following three situations.

- There is a unique solution, when the three planes represented by the equations intersect at a unique point, as illustrated in Figure 8.

  For example, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} - z &= 0 \end{aligned}$$

  has the unique solution $x = 0$, $y = 1$, $z = 0$, corresponding to the unique point of intersection $(0, 1, 0)$ of the three planes in $\mathbb{R}^3$.

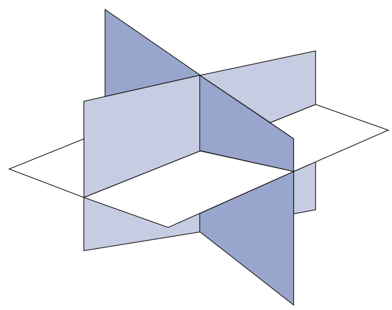

Figure 8 Three planes intersecting in a unique point

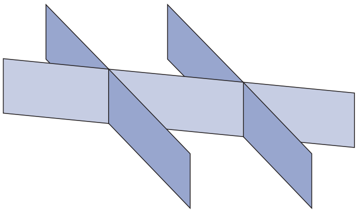

Figure 9 Three planes, two of which are parallel

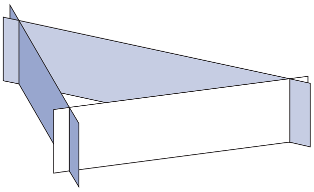

Figure 10 Three planes intersecting in pairs forming a prism

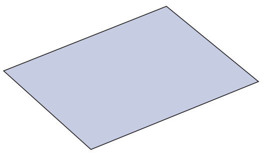

Figure 11 Three coincident planes

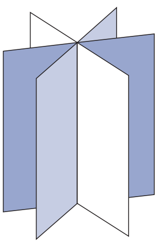

Figure 12 Three planes intersecting in a line

- There is no solution, when two (or three) of the planes represented by the equations are parallel, or when the three planes form a triangular prism, as illustrated in Figures 9 and 10, respectively.

  For example, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \\ x + y - z &= 0 \end{aligned}$$

  represents three planes in $\mathbb{R}^3$, the first two of which are parallel, and so the system has no solutions.

  Similarly, the system

  $$\begin{aligned} x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} + z &= 1 \\ - y + z &= 1 \end{aligned}$$

  has no solutions: the planes in $\mathbb{R}^3$ represented by the three equations intersect in pairs, forming a triangular prism, and so there are no points common to *all three* planes.

- There are infinitely many solutions, when the three planes that the equations represent intersect either in a plane or in a line, as illustrated in Figures 11 and 12, respectively.

  For example, the system

  $$\begin{aligned} x + y + z &= 1 \\ -x - y - z &= -1 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

  has infinitely many solutions, as the three equations all represent the same plane in $\mathbb{R}^3$: the equations are multiples of one another. Each set of values for x, y and z satisfying $x + y + z = 1$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = -1$, $y = 3$, $z = -1$.

  Similarly, the system

  $$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x + y - z &= 1 \end{aligned}$$

  has infinitely many solutions: the planes in $\mathbb{R}^3$ represented by the three equations intersect in a line. The z-coordinate of each point on this line is zero, and so the line lies in the (x, y)-plane. Each set of values for x, y and z satisfying $x + y = 1$ and $z = 0$ is a solution to this system, such as $x = 1$, $y = 0$, $z = 0$ and $x = -5$, $y = 6$, $z = 0$.

1.2 Systems in n unknowns

The equations for a line in $\mathbb{R}^2$ and a plane in $\mathbb{R}^3$ are *linear equations* in two and three unknowns, respectively. Similarly, an equation of the form

$$ax + by + cz + dw = e$$

is a linear equation in the four unknowns x, y, z and w, where $a, \ldots, e$ are real numbers, and a, b, c and d are not all zero. In general, we can define a linear equation in any number of unknowns.

Definitions

An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are real numbers, and $a_1, \ldots, a_n$ are not all zero, is a **linear equation** in the n **unknowns** $x_1, x_2, \ldots, x_n$. The numbers a_i are the **coefficients**, and b is the **constant term**.

A linear equation has no terms that are products of unknowns, such as x_1^2 or x_1x_4.

Exercise C1

Which of the following are linear equations in the five unknowns $x_1, \ldots, x_5$?

(a) $x_1 + 3x_2 - x_3 - 5x_4 - 2x_5 = 0$ (b) $x_1 - x_2 + 2x_3x_4 + 3x_5 = 4$

(c) $5x_2 - x_5 = 2$ (d) $a_1x_1 + a_2x_2^2 + \cdots + a_5x_5^5 = b$

We write a system of linear equations, or more precisely *a general system of m linear equations in n unknowns*, as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

The numbers b_i are the constant terms, the variables x_j are the unknowns and the numbers a_{ij} are the coefficients. We use the double subscript ij to show that a_{ij} is the coefficient of the jth unknown in the ith equation. The number m of equations need not be the same as the number n of unknowns.

A solution of a system of linear equations is a list of values for the unknowns that simultaneously satisfy each of the equations. In solving a system, we look for *all* the solutions – you have already seen that some systems have infinitely many solutions.

Definitions

The values $x_1 = c_1$, $x_2 = c_2$, ..., $x_n = c_n$ are a **solution** of a system of m linear equations in n unknowns, denoted by $x_1, \ldots, x_n$, if these values simultaneously satisfy all m equations of the system. The **solution set** of the system is the set of all the solutions.

For example, you saw earlier that the system

$$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} - z &= 0, \end{aligned} \tag{1}$$

has the unique solution $x = 0$, $y = 1$, $z = 0$ corresponding to the unique point of intersection $(0, 1, 0)$ of the three planes represented by these equations. We can write the solution set of this system as the set $\{(0, 1, 0)\}$, which has just one member.

You also saw that the system

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \\ x + y - z &= 0, \end{aligned} \tag{2}$$

has no solutions, so its solution set is the empty set.

Definitions

A system of linear equations is **consistent** when it has at least one solution, and **inconsistent** when it has no solutions.

The system (1) is consistent, and the system (2) is inconsistent.

When a system of linear equations has infinitely many solutions, we can write down a general solution from which all solutions can be found as follows.

You saw earlier that the solutions of the system

$$\begin{aligned} x + y + z &= 1 \\ x + y \phantom{{}+z} &= 1 \\ x + y - z &= 1 \end{aligned}$$

are the sets of values for x, y and z satisfying $x + y = 1$ and $z = 0$. The unknowns x and y are related by the equation $x + y = 1$, which we can rewrite as $y = 1 - x$. Thus for each real parameter k assigned to the unknown x, we have a corresponding value $1 - k$ for the unknown y. We write this general solution as

$$x = k, \quad y = 1 - k, \quad z = 0, \quad \text{where } k \in \mathbb{R}.$$

To highlight the connection between the solutions of the system and the intersection of the planes in $\mathbb{R}^3$, we can write the solution set as a set of points in $\mathbb{R}^3$:

$$\{(k, 1-k, 0) \in \mathbb{R}^3 : k \in \mathbb{R}\},$$

which we usually abbreviate to

$$\{(k, 1-k, 0) : k \in \mathbb{R}\}.$$

Note that the order of the unknowns x, y, then z matters: the triples $(1, 0, 0)$ and $(0, 1, 0)$ correspond to different solutions. We could have assigned parameters differently and obtained alternative ways of writing down the solution set. For example, if we assign the real parameter p to the unknown y and rewrite the equation $x + y = 1$ as $x = 1 - y$, we get

$$\{(1-p, p, 0) : p \in \mathbb{R}\}.$$

Homogeneous systems

In the following systems of linear equations, the constant terms are all zero:

$$\begin{aligned} 2x + 3y &= 0 \\ x - y &= 0, \end{aligned} \tag{3}$$

$$\begin{aligned} x - y - z &= 0 \\ 2x + y - z &= 0 \\ -x + y + z &= 0. \end{aligned} \tag{4}$$

Such systems are called *homogeneous*.

Definitions

A **homogeneous** system of linear equations is a system of linear equations in which each constant term is zero.

A system containing at least one non-zero constant term is a **non-homogeneous** system.

If we substitute $x = 0$, $y = 0$ into system (3), and $x = 0$, $y = 0$, $z = 0$ into system (4), then all the equations are satisfied. These solutions are called *trivial*.

Definitions

The **trivial** solution to a system of homogeneous linear equations is the solution in which each unknown is equal to zero.

A solution with at least one non-zero unknown is a **non-trivial** solution.

A homogeneous system always has at least the trivial solution, and is therefore always consistent, whereas non-homogeneous systems have only non-trivial solutions or may be inconsistent.

Exercise C2

Write down a general homogeneous system of m linear equations in n unknowns, and show that the solution set contains the trivial solution.

Returning to system (4), we see that there are other solutions, unlike system (3) which has no non-trivial solutions. For example, $x = 2$, $y = -1$, $z = 3$ is a solution to system (4). In fact, this system has an infinite solution set because the first and third equations are multiples of one another. Geometrically, the three planes represented by these equations intersect in a line. Figure 7 illustrates this situation, as the planes represented by the first and third equations coincide. The solution set can be written as $\{(2k, -k, 3k) : k \in \mathbb{R}\}$.

Number of solutions

In Subsection 1.1 you saw that when $m \leq n \leq 3$, a system of m equations in n unknowns has a solution set which either

- contains exactly one solution,
- is empty, or
- contains infinitely many solutions.

(When $m = n = 1$ we have one equation of the form $ax = b$, which has a unique solution.)

In fact, as you will see in Unit C3 *Linear transformations*, the solution set of a system of m linear equations in n unknowns has one of these forms, for any natural numbers m and n.

We observed earlier that two non-parallel planes in $\mathbb{R}^3$ intersect either in a line or in a plane, so cannot intersect at a unique point. A consistent system of two linear equations in three unknowns therefore has an infinite solution set. In general, a consistent system of m equations in n unknowns, with $m < n$, has insufficient constraints on the unknowns to determine them uniquely; that is, it has an infinite solution set.

1.3 Solving systems

We now introduce a systematic method for solving systems of linear equations. This method is called **Gauss–Jordan elimination**. It entails successively transforming a system into simpler systems, in such a way that the solution set remains unchanged. The process ends when the solutions can be determined easily. You will meet this method again in Section 2, where you will use *matrices* to represent systems of linear equations. A strategy for solving systems of linear equations using Gauss–Jordan elimination is given there.

The Gauss–Jordan elimination method was introduced by the geodesist Wilhelm Jordan (1842–1899) in the third edition of his *Handbuch der Vermessungskunde* (*Handbook of Surveying*) in 1888. In the same year the rather more obscure Luxembourg mathematician turned abbot Bernard Isidore Clasen (1829–1902) independently described the method, but his work did not become widely known. The method's association with Carl Friedrich Gauss (1777–1855) is due to the fact that it can be regarded as a modification of the method of Gaussian elimination. Wilhelm Jordan is not to be confused with the algebraist Camille Jordan (1838–1922).

Wilhelm Jordan

The idea of Gauss–Jordan elimination is to reduce the number of unknowns in each equation. In general, we use the first equation to eliminate the first unknown from all the other equations, then use the second equation to eliminate another unknown (usually the second) from all the other equations, and so on. The actual order in which the unknowns are eliminated is flexible; however, it is sensible, at least initially, to proceed in order to avoid making mistakes.

To avoid confusion when applying this method, we label the current equations $\mathbf{r}_1$, $\mathbf{r}_2$, and so on. This notation will be used in Section 2 where we transform *rows* of matrices, hence the choice of the letter $\mathbf{r}$.

We can then write down how we are transforming the preceding system to obtain the current (simpler) system. We use the symbol $\leftrightarrow$ ('interchanges with') to indicate that two equations are to be interchanged; for example, $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ means that the first and second equations are interchanged. We use the symbol $\rightarrow$ ('goes to') to show how an equation is to be transformed. For example, $\mathbf{r}_2 \rightarrow \mathbf{r}_2 + \mathbf{r}_1$ means that the second equation of the system is transformed by adding the first equation to it.

We start by illustrating this method with a system of two linear equations in two unknowns. Although this method is not the simplest way of solving this particular system, it proves very useful in solving more complicated systems. It is important that the operations we perform do not alter the solution set of the system.

Worked Exercise C1

Solve the following system of two linear equations in two unknowns.

$$\begin{aligned} 2x + 4y &= 10 \\ 4x + y &= 6 \end{aligned}$$

Solution

We aim to simplify the system by eliminating the unknown y, or y-term, from the first equation and the unknown x, or x-term, from the second; that is, we aim to obtain equations of the form $x = *$, $y = *$, where the asterisks denote numbers to be determined.

We label the two equations of the system.

$$\begin{array}{ll} \mathbf{r}_1 & 2x + 4y = 10 \\ \mathbf{r}_2 & 4x + y = 6 \end{array}$$

We simplify the first equation.

We divide it through by 2, so that the coefficient of x is equal to 1.

$$\begin{array}{ll} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1 & x + 2y = 5 \\ & 4x + y = 6 \end{array}$$

At each step, we relabel (implicitly) the equations of the current system. These two equations therefore become the *new* $\mathbf{r}_1$ and $\mathbf{r}_2$.

We then eliminate the x-term in the second equation.

We subtract 4 times the first equation from the second.

$$\begin{array}{ll} & x + 2y = 5 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_1 & - 7y = -14 \end{array}$$

We now simplify the second equation of this new system.

We divide it through by -7; this yields a system which already looks less complicated than the original system, but has the same solution set.

$$\begin{array}{ll} & x + 2y = 5 \\ \mathbf{r}_2 \to -\frac{1}{7}\mathbf{r}_2 & y = 2 \end{array}$$

Next we eliminate the y-term from the first equation.

We subtract twice the second equation from the first.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & x = 1 \\ & y = 2 \end{array}$$

We conclude that there is a unique solution: $x = 1$, $y = 2$.

As a check: we substitute $x = 1$ and $y = 2$ in the original system, using the abbreviation LHS for the left-hand side of the original equations and RHS for the right-hand side:

$$\begin{aligned} \text{LHS} &= (2 \times 1) + (4 \times 2) = 10 = \text{RHS}, \quad \checkmark \\ \text{LHS} &= (4 \times 1) + (1 \times 2) = 6 = \text{RHS}. \quad \checkmark \end{aligned}$$

The steps we performed in the worked exercise above involve either multiplying (or dividing) an equation by a non-zero number, or changing one equation by adding (or subtracting) a multiple of another. Neither of these operations alters the solution set of the system. Changing the order in which we write down the equations also does not alter the solution set of the system. These are the three operations, called *elementary operations*, that we perform to simplify a system of linear equations when using the method of Gauss–Jordan elimination.

Elementary operations

The following operations do not change the solution set of a system of linear equations.

1. Interchange two equations.
2. Multiply an equation by a non-zero number.
3. Change one equation by adding to it a multiple of another.

Operation 2 includes division by a non-zero number, and operation 3 includes subtracting a multiple of one equation from another.

In symbols we represent these three elementary operations by

$$\mathbf{r}_i \leftrightarrow \mathbf{r}_j, \quad \mathbf{r}_i \to \alpha\mathbf{r}_i, \quad \text{and} \quad \mathbf{r}_i \to \mathbf{r}_i + \beta\mathbf{r}_j,$$

respectively, where α, β are non-zero numbers.

Exercise C3

Perform elementary operations, as in Worked Exercise C1, to solve the following system of two linear equations in two unknowns.

$$\begin{aligned} x + y &= 4 \\ 2x - y &= 5 \end{aligned}$$

We now solve a system of three linear equations in three unknowns. We use elementary operations to try to reduce the system to the following form, where again the asterisks denote numbers to be determined.

$$\begin{aligned} x &= * \\ y &= * \\ z &= * \end{aligned}$$

Worked Exercise C2

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + y + 2z &= 3 \\ 2x + 2y + 3z &= 5 \\ x - y \phantom{{}+3z} &= 5 \end{aligned}$$

Solution

We label the three equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y + 2z = 3 \\ \mathbf{r}_2 & 2x + 2y + 3z = 5 \\ \mathbf{r}_3 & x - y \phantom{{}+2z} = 5 \end{array}$$

We eliminate the x-term from the second and third equations: we subtract twice the first equation from the second, and then subtract the first from the third.

$$\begin{array}{ll} & x + y + 2z = 3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & -z = -1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & -2y - 2z = 2 \end{array}$$

We now have no y-term in the second equation, and so cannot use this equation to eliminate the y-term from the first and third equations. We also cannot use the first equation to eliminate the y-term from the third equation, as this would reintroduce an x-term. However, we can use the third equation to eliminate the y-term from the first equation. To keep the terms in order we interchange the second and third equations – this is not strictly necessary, but helps keep things in order.

$$\begin{array}{ll} & x + y + 2z = 3 \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 & -2y - 2z = 2 \\ & -z = -1 \end{array}$$

We simplify this new second equation by dividing it through by -2.

$$\begin{array}{ll} & x + y + 2z = 3 \\ \mathbf{r}_2 \to -\tfrac{1}{2}\mathbf{r}_2 & y + z = -1 \\ & -z = -1 \end{array}$$

We eliminate the y-term from the first equation by subtracting the second equation from the first.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x + z = 4 \\ & y + z = -1 \\ & -z = -1 \end{array}$$

We simplify the third equation by multiplying it through by -1.

$$\begin{array}{lrcl} & x \quad + z &=& 4 \\ & y + z &=& -1 \\ \mathbf{r}_3 \to -\mathbf{r}_3 & z &=& 1 \end{array}$$

Finally, we eliminate the z-term from the first and second equations by subtracting the third equation from both the first and second equations.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_3 & x = 3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 & y = -2 \\ & z = 1 \end{array}$$

We conclude that there is a unique solution: $x = 3$, $y = -2$, $z = 1$.

As a check: we substitute $x = 3$, $y = -2$ and $z = 1$ into the original system:

$$\begin{aligned} \text{LHS} &= (1 \times 3) + (1 \times (-2)) + (2 \times 1) = 3 = \text{RHS}, \checkmark \\ \text{LHS} &= (2 \times 3) + (2 \times (-2)) + (3 \times 1) = 5 = \text{RHS}, \checkmark \\ \text{LHS} &= (1 \times 3) - (1 \times (-2)) + (0 \times 1) = 5 = \text{RHS}. \checkmark \end{aligned}$$

Exercise C4

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + y - z &= 8 \\ 2x - y + z &= 1 \\ -x + 3y + 2z &= -8 \end{aligned}$$

Each system solved so far in this subsection has a unique solution. We now show how to apply the method to a system that does not have a unique solution.

It is not usually possible to reduce a system with an infinite solution set to one where each equation contains just one unknown. This is illustrated by the following worked exercise.

Worked Exercise C3

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + 2y \phantom{{}-z} &= 0 \\ y - z &= 2 \\ x + y + z &= -2 \end{aligned}$$

Solution

We label the three equations and apply elementary operations to simplify the system.

$$\begin{array}{lrcrcrcl}
\mathbf{r}_1 & x & + & 2y & & & = & 0 \\
\mathbf{r}_2 & & & y & - & z & = & 2 \\
\mathbf{r}_3 & x & + & y & + & z & = & -2
\end{array}$$

We eliminate the x-term from $\mathbf{r}_3$ using $\mathbf{r}_1$.

$$\begin{array}{lrcrcrcl}
 & x & + & 2y & & & = & 0 \\
 & & & y & - & z & = & 2 \\
\mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & & & -y & + & z & = & -2
\end{array}$$

We eliminate the y-terms from $\mathbf{r}_1$ and $\mathbf{r}_3$ using $\mathbf{r}_2$.

$$\begin{array}{lrcrcrcl}
\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & x & & & + & 2z & = & -4 \\
 & & & y & - & z & = & 2 \\
\mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 & 0x & + & 0y & + & 0z & = & 0
\end{array}$$

The current $\mathbf{r}_3$ equation gives no constraints on x, y and z: any values for x, y and z satisfy it.

If we were to try to use equation $\mathbf{r}_2$ to eliminate the z-term from $\mathbf{r}_1$, we would introduce a y-term. Similarly, using equation $\mathbf{r}_1$ to eliminate the z-term from the equation $\mathbf{r}_2$ would reintroduce an x-term.

There are insufficient constraints on the unknowns to determine them uniquely; so the system has an infinite solution set.

We have two equations, one ($x = -4 - 2z$) relating the unknowns x and z, and the other ($y = 2 + z$) relating y and z.

As each equation involves a z-term, we can choose any value we wish for z and use the equations to find the corresponding values for x and y in terms of this value for z. We set z equal to the real parameter k to get a general solution.

We write the general solution as

$$x = -4 - 2k, \quad y = 2 + k, \quad z = k, \quad k \in \mathbb{R}.$$

In the worked exercise above the equation $\mathbf{r}_3$ was written as $0x + 0y + 0z = 0$ to highlight the fact that all the coefficients are zero – in future we will simply write the equivalent equation $0 = 0$. In this case, the original equation $\mathbf{r}_3$ did not give rise to any additional constraints not already given by $\mathbf{r}_1$ and $\mathbf{r}_2$.

Whenever the simplification results in an equation $0 = 0$, we have, in effect, reduced the number of equations. We simplify the remaining equations as far as possible, in order to determine the solution set.

Exercise C5

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + 3y - z &= 4 \\ -x + 2y - 4z &= 6 \\ x + 2y \phantom{{}-4z} &= 2 \end{aligned}$$

We now try to solve an inconsistent system.

Worked Exercise C4

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + 2y + 4z &= 6 \\ y + z &= 1 \\ x + 3y + 5z &= 10 \end{aligned}$$

Solution

We label the three equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + 2y + 4z = 6 \\ \mathbf{r}_2 & y + z = 1 \\ \mathbf{r}_3 & x + 3y + 5z = 10 \end{array}$$

We eliminate the x-term from $\mathbf{r}_3$ using $\mathbf{r}_1$.

$$\begin{array}{ll} & x + 2y + 4z = 6 \\ & y + z = 1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & y + z = 4 \end{array}$$

Comparing $\mathbf{r}_2$ and $\mathbf{r}_3$, we can conclude at this point that the system is inconsistent or we can carry out one further step to eliminate the y-terms from $\mathbf{r}_1$ and $\mathbf{r}_3$ using $\mathbf{r}_2$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & x \phantom{{}+y} + 2z = 4 \\ & \phantom{x+{}} y + z = 1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_2 & 0 = 3 \end{array}$$

Concentrating on the current $\mathbf{r}_3$ equation $(0 = 3)$, we see that there are no values of x, y and z that satisfy it. This system has no solutions.

This system of linear equations is inconsistent: the solution set is the empty set.

Whenever the simplification results in an equation $0 = *$, where the asterisk $*$ denotes a non-zero number, we have an inconsistent system, since such an equation has no solutions. There is no point in simplifying the remaining equations further. As indicated in Worked Exercise C4, inconsistency of the system could have been inferred at the penultimate stage, as the equations $y + z = 1$ and $y + z = 4$ form an inconsistent system.

Exercise C6

Solve the following system of three linear equations in three unknowns.

$$\begin{aligned} x + y + z &= 6 \\ -x + y - 3z &= -2 \\ 2x + y + 3z &= 6 \end{aligned}$$

1.4 Applications

Systems of linear equations frequently arise when we use mathematics to solve problems from both within mathematics and outside it.

The following worked exercise illustrates how linear equations can be used to find the equation of a plane through three given points.

Worked Exercise C5

Determine the equation of the plane that contains the three points $(1, 3, 1)$, $(1, 5, 2)$ and $(2, 2, 1)$.

Solution

Let the equation of the plane be

$$ax + by + cz = d,$$

where a, b, c and d are real, and a, b and c are not all zero.

Each of the points is contained in the plane, and so each set of coordinates satisfies this equation.

We do not specify a value for d at this point because we can multiply or divide the equation through by a constant without affecting the plane it represents.

Substituting the points into the equation gives a system of three linear equations in the unknowns a, b and c. We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & a + 3b + c = d \\ \mathbf{r}_2 & a + 5b + 2c = d \\ \mathbf{r}_3 & 2a + 2b + c = d \end{array}$$

We eliminate the a-term from $\mathbf{r}_2$ and $\mathbf{r}_3$ using $\mathbf{r}_1$.

$$\begin{array}{ll} & a + 3b + c = d \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 & \quad 2b + c = 0 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 & \quad -4b - c = -d \end{array}$$

We simplify $\mathbf{r}_2$.

$$\begin{array}{ll} & a + 3b + \ \ c = d \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 & \qquad b + \frac{1}{2}c = 0 \\ & \quad -4b - \ \ c = -d \end{array}$$

We eliminate the b-term from $\mathbf{r}_1$ and $\mathbf{r}_3$ using $\mathbf{r}_2$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & a \quad - \frac{1}{2}c = d \\ & \quad b + \frac{1}{2}c = 0 \\ \mathbf{r}_3 \to \mathbf{r}_3 + 4\mathbf{r}_2 & \qquad\quad c = -d \end{array}$$

We eliminate the c-term from $\mathbf{r}_1$ and $\mathbf{r}_2$ using $\mathbf{r}_3$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 + \frac{1}{2}\mathbf{r}_3 & a = \frac{1}{2}d \\ \mathbf{r}_2 \to \mathbf{r}_2 - \frac{1}{2}\mathbf{r}_3 & b = \frac{1}{2}d \\ & c = -d \end{array}$$

We conclude that this system has a unique solution (in terms of d): $a = \frac{1}{2}d,\ b = \frac{1}{2}d,\ c = -d.$

We substitute these expressions into the equation of the plane to get

$$\tfrac{1}{2}dx + \tfrac{1}{2}dy - dz = d.$$

Multiplying through by 2 and dividing through by d yields a simpler equation for the plane:

$$x + y - 2z = 2.$$

As a check: we substitute the coordinates of each of the three points into this equation for the plane

LHS $= (1 \times 1) + (1 \times 3) - (2 \times 1) = 2 =$ RHS, ✓
LHS $= (1 \times 1) + (1 \times 5) - (2 \times 2) = 2 =$ RHS, ✓
LHS $= (1 \times 2) + (1 \times 2) - (2 \times 1) = 2 =$ RHS. ✓

Exercise C7

Determine the equation of the plane that contains the three points $(1, 0, 2)$, $(0, 3, 4)$ and $(1, 1, 3)$.

The final exercise in this section uses systems of linear equations to solve a different type of problem. The idea is to use the information given to write down two linear equations that simultaneously hold, and then to solve these to answer the question.

Exercise C8

The sum of the ages of my sister and my brother is 40 years. My brother is 12 years older than my sister. How old is my sister?

Because Gauss–Jordan elimination is a systematic method for solving systems of linear equations, it is straightforward to automate. Hence large systems of linear equations involving many variables can be easily solved using computers. Such systems are used in some methods of weather forecasting, as well as systems of non-linear equations. Gauss–Jordan elimination also arises in coding theory, which underpins digital communication and data transmission.

2 Row-reduction

In this section you will see how the method of Gauss–Jordan elimination can be applied using matrices, and that it can be formalised into a strategy that can be followed step by step. This method makes it easy to solve even quite large systems of linear equations. It involves a technique (*row-reduction*) that will be useful in another context later in this unit when we look at inverses of matrices.

2.1 Augmented matrices

We begin by using *matrices* as an abbreviated notation for a system of linear equations.

A **matrix** is simply a rectangular array of objects, usually numbers, enclosed in brackets; in this module we use round brackets for matrices, although some texts use square ones.

The objects in a matrix are called its **entries**. The entries along a horizontal line form a **row**, and those down a vertical line form a **column**. A matrix with m rows and n columns is an $m \times n$ matrix, and we say that it is a matrix of **size** $m \times n$.

A **zero row** of a matrix is a row comprising entirely of zeros, and a **non-zero row** has at least one non-zero entry. The first non-zero entry in a row (reading from left to right) of a matrix is the **leading entry** of that row; when such an entry in a row is the number 1, it is called a **leading 1**.

Here are some examples of matrices with some entries highlighted as explained below:

$$\begin{pmatrix} 2 & -7 \\ -1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3.17 & 2.23 & 7.05 & 0.00 \\ 4.88 & 1.71 & 1.72 & 5.55 \end{pmatrix}, \quad \begin{pmatrix} 3 & 7 & 12 \\ 0 & 1 & 8 \\ 0 & 0 & -5 \end{pmatrix}.$$

The entries in the first row of the first matrix above are 2 and -7; the entries in the second column of the second matrix above are 2.23 and 1.71; the 1 in the second row of the third matrix is a leading 1, and the -5 in the third row of this matrix is a leading entry.

We can abbreviate a system of linear equations by writing its coefficients and constants in the form of a matrix. For example, the system

$$\begin{aligned} 4x + y &= -7 \\ x - 3y &= 0 \end{aligned}$$

can be abbreviated as

$$\left(\begin{array}{cc|c} 4 & 1 & -7 \\ 1 & -3 & 0 \end{array}\right).$$

It is helpful to draw the vertical line separating the coefficients of the unknowns on the left-hand sides of the equations from the constants on the right-hand sides.

In general, the system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is abbreviated as the matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

This matrix is called the **augmented matrix** of the system. The word *augmented* reflects the fact that it is made up of a matrix formed by the coefficients of the unknowns on the left-hand sides of the equations, *augmented* by a matrix (or column vector) formed by the constants on the right-hand sides. Later, we will sometimes consider these two matrices separately.

In the augmented matrix each row corresponds to an equation, and each column (except the last) corresponds to an unknown, in the sense that it contains all the coefficients of that unknown from the various equations. The last column corresponds to the right-hand sides of the equations.

Worked Exercise C6

Write down the augmented matrix of the following system of linear equations.

$$\begin{aligned} x \phantom{{}+2y} + 10z &= 5 \\ 3x + y - 4z &= -1 \\ 4x - 2y + 6z &= 1 \end{aligned}$$

Solution

Before writing down the augmented matrix of a system of linear equations, we must ensure that the unknowns appear in the same order in each equation, with gaps left for 'missing' unknowns (that is, unknowns whose coefficient is 0).

The augmented matrix of the system is

$$\left(\begin{array}{rrr|r} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & -2 & 6 & 1 \end{array}\right).$$

Worked Exercise C7

Write down the system of linear equations corresponding to the following augmented matrix, given that the unknowns are, in order, x_1, x_2.

$$\left(\begin{array}{rr|r} 1 & -2 & 5 \\ 0 & 1 & 9 \\ 4 & 3 & 0 \end{array}\right)$$

Solution

The corresponding system is

$$\begin{aligned} x_1 - 2x_2 &= 5 \\ x_2 &= 9 \\ 4x_1 + 3x_2 &= 0. \end{aligned}$$

Exercise C9

(a) Write down the augmented matrix of the following system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 \phantom{{}+3x_3} &= -7 \\ x_2 + 3x_3 &= 0 \\ -3x_2 + x_3 &= 3 \end{aligned}$$

(b) Write down the system of linear equations corresponding to the following augmented matrix, given that the unknowns are, in order, x, y, z, w.

$$\left(\begin{array}{rrrr|r} 2 & 3 & 0 & 7 & 1 \\ 0 & 1 & -7 & 0 & -1 \\ 1 & 0 & 3 & -1 & 2 \end{array}\right)$$

2.2 Elementary row operations

When you used Gauss–Jordan elimination to solve a system of linear equations in Section 1, you worked directly with the system itself; but it is often easier to apply the same method to its abbreviated form, the augmented matrix. The three elementary operations on the equations of the system correspond exactly to three equivalent operations on the rows of its augmented matrix.

Recall that the three elementary operations are as follows.

1. Interchange two equations.
2. Multiply an equation by a non-zero number.
3. Change one equation by adding to it a multiple of another.

These correspond to the following operations on the rows of the augmented matrix.

Elementary row operations

1. Interchange two rows.
2. Multiply a row by a non-zero number.
3. Change one row by adding to it a multiple of another.

We call these operations the **elementary row operations** of types 1, 2 and 3, respectively.

The next worked exercise shows a system of linear equations solved by Gauss–Jordan elimination. In Worked Exercise C2 we solved this system by performing elementary operations on the system itself; here we perform the corresponding elementary row operations on the augmented matrix of the system. You can see that here we have less to write down at each stage.

In this worked exercise, and elsewhere, we use the same notation for elementary row operations as we use for elementary operations ($\mathbf{r}_i \leftrightarrow \mathbf{r}_j$, and so on).

Worked Exercise C8

Solve the following system of linear equations.

$$\begin{aligned} x + y + 2z &= 3 \\ 2x + 2y + 3z &= 5 \\ x - y \phantom{{}+3z} &= 5 \end{aligned}$$

Solution

We perform a sequence of elementary row operations on the augmented matrix of the system.

The idea is to transform the augmented matrix into one of a system with the same solution set but whose solution set is easy to write down.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{2}\mathbf{r}_2 \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\mathbf{r}_3 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The corresponding system is

$$\begin{aligned} x \qquad &= 3 \\ y \quad &= -2 \\ z &= 1. \end{aligned}$$

The unique solution is $x = 3$, $y = -2$, $z = 1$.

It is important to appreciate the following point about elementary row operations.

When a sequence of elementary row operations is performed on a matrix, each row operation in the sequence produces a new matrix, and the following row operation is then performed on that new matrix. For example, the working for the first two row operations in the solution to the

worked exercise above should, strictly, be set out as follows.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{array}\right)$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \\ \end{array} \quad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 5 \end{array}\right)$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right)$$

However, we often perform two or more row operations in one step, to save time. Whenever we do this, we must ensure that when a row is changed by one of these row operations, the *new version* of that row is used when performing later row operations.

The easiest way to avoid difficulties is to perform two or more row operations in one step *only if none of these row operations changes a row that is then used by another of these row operations.* The above row operations $\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1$ and $\mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1$ meet this criterion: the first changes only row 2, and the second does not involve row 2. In this module we perform two or more row operations in one step only if they meet this criterion.

Row-sum check

We end this subsection by describing a simple checking method that can be useful for picking up arithmetical errors when we perform a sequence of elementary row operations on a matrix by hand.

To apply this method, we proceed as follows. To the right of each row of the initial matrix, we write down the sum of the entries in that row.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{array}\right) \quad \begin{array}{rl} 7 & (= 1 + 1 + 2 + 3) \\ 12 & (= 2 + 2 + 3 + 5) \\ 5 & (= 1 - 1 + 5) \end{array}$$

From then on, when performing elementary row operations, we treat this 'check column' of numbers as if it were an extra column of the matrix, and perform the row operations on it. So the first step of the calculation in the solution to Worked Exercise C8 above would look as follows.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|r} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{array}\right) \quad \begin{array}{rl} 7 & \\ -2 & (= 12 - 2 \times 7) \\ -2 & (= 5 - 7) \end{array}$$

At each step in the calculation, each entry in this extra column should still be the sum of the entries in the corresponding row. If this is not the case, then an error has been made.

2.3 Solving linear equations systematically

We now describe a systematic method for solving a system of linear equations by Gauss–Jordan elimination. The method involves performing elementary row operations on the augmented matrix of the system. In fact you have already seen this method in action, in Worked Exercise C8. Here we detail the sequence of steps involved, setting out a general strategy.

The strategy involves writing down the augmented matrix of the system of equations and then performing a sequence of elementary row operations that reduces it to a simpler form called *row-reduced form*. The system of equations corresponding to this row-reduced matrix has the same solution set as the original system but with the new system it is easy to work out what the solution set is. The process of reducing the matrix to row-reduced form is referred to as *row-reduction*. We start by describing what a row-reduced matrix looks like.

Row-reduced matrices

Here is an example of a row-reduced matrix.

$$\begin{pmatrix} 0 & 1 & 6 & 0 & -5 & 3 & 0 & 0 & 17 \\ 0 & 0 & 0 & 1 & 3 & 24 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries of a row-reduced matrix have a staircase form which we have emphasised with a black line. All the entries below the staircase must be 0; the left-most entry on each line above the staircase must be a 1 and all the other entries in that column must be 0. The other entries above the staircase can be any numbers.

The general form of a row-reduced matrix is illustrated below, where the entries not in a column containing a leading 1 are indicated with asterisks, and the fact that all the entries below the staircase are zero is indicated by the large zero.

$$\begin{pmatrix} 1 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & 1 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * \\ & & & & 1 & * \cdots * & 0 & * \cdots * \\ & 0 & & & & & 1 & * \cdots * \\ & & & & & & & \end{pmatrix}$$

We can describe a row-reduced matrix more precisely by specifying that it must satisfy certain properties as in the definition below.

Definition

A **row-reduced matrix** is a matrix satisfying the following four properties.

1. Any zero rows are at the bottom of the matrix.
2. Each non-zero row has a leading 1.
3. Each leading 1 is to the right of the leading 1 in the row above.
4. Each leading 1 is the only non-zero entry in its column.

Property 3 gives a row-reduced matrix its staircase form, and property 4 ensures that the entries above and below a leading 1 are all 0.

Here are some more examples of row-reduced matrices:

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 7 & 0 & 23 & 0 \\ 0 & 1 & \frac{3}{4} & -8 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, none of the following matrices are row-reduced:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 5 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first matrix is not row-reduced as it has neither property 1 nor property 2. The second is not row-reduced as it does not have property 3; the third does not have property 4.

Exercise C10

Which of the following are row-reduced matrices?

(a) $\begin{pmatrix} 0 & 1 & 7 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Finding solutions from row-reduced form

Suppose we have been given a system of linear equations, and that we have written down its augmented matrix and performed a sequence of elementary row operations to reduce it to row-reduced form. We will describe these operations in detail shortly, but first we will consider how to find all the solutions from the row-reduced form.

Unique solution

Consider for example this row-reduced augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Suppose the unknowns are x_1, x_2 and x_3. Then the system of equations corresponding to this matrix is as follows.

$$\begin{aligned} x_1 \qquad\qquad &= 8 \\ x_2 \qquad &= 3 \\ x_3 &= -1 \end{aligned}$$

This system is already in solved form, and we can immediately write down the unique solution:

$$x_1 = 8,\ x_2 = 3,\ x_3 = -1.$$

Unique solution

Whenever the original system of equations has a unique solution it can be written down directly from the row-reduced matrix.

No solution

Now consider this row-reduced augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

We write down the system of equations corresponding to the matrix.

$$\begin{aligned} x_1 - 6x_2 \qquad &= 0 \\ x_3 &= 0 \\ 0 &= 1 \end{aligned}$$

This time we find that one of the equations is $0 = 1$. This equation cannot hold, so it follows that the system of equations has no solutions; that is, the equations are inconsistent.

No solution

Whenever the original system of equations is inconsistent, row-reducing the augmented matrix yields a system that includes the equation $0 = 1$.

Infinitely many solutions

Now consider this row-reduced augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & -4 & 2 \end{array}\right).$$

We can write down the system of equations corresponding to this row-reduced matrix, but this time the system does not immediately give us a solution nor does it include the equation $0 = 1$.

$$\begin{aligned} x_1 \quad\;\; + 6x_3 &= 7 \\ x_2 - 4x_3 &= 2 \end{aligned}$$

In this case we rearrange each equation so that everything except the first term on the left is moved over to the right-hand side. The effect of this is to express each **leading unknown**, that is, each unknown that corresponds to a column containing a leading 1, in terms of the other unknowns, the **non-leading unknowns**. Here, x_1 and x_2 are leading unknowns and x_3 is the only non-leading unknown.

$$\begin{aligned} x_1 &= 7 - 6x_3 \\ x_2 &= 2 + 4x_3 \end{aligned}$$

Having expressed the two leading unknowns, x_1 and x_2, in terms of the non-leading unknown x_3 we can then choose any value we like for x_3 and the equations give us the corresponding values of x_1 and x_2. So the system has infinitely many solutions – one for each choice of value of x_3. If we set $x_3 = k$ ($k \in \mathbb{R}$), say, and substitute this into the expressions for x_1 and x_2, then we have all the unknowns expressed in terms of the parameter k. The general solution of the system is therefore:

$$\begin{aligned} x_1 &= 7 - 6k \\ x_2 &= 2 + 4k \\ x_3 &= k \quad (k \in \mathbb{R}). \end{aligned}$$

Infinitely many solutions

Whenever the original system of equations has infinitely many solutions, the general solution can be determined by setting the non-leading unknowns equal to parameters and expressing all the unknowns in terms of these parameters.

As we noted in Subsection 1.2, a system of linear equations must have one of these three possibilities: a unique solution, no solution, or infinitely many solutions.

Worked Exercise C9

Solve the system corresponding to the following row-reduced augmented matrix. Assume that the unknowns are x_1, x_2, x_3, x_4, x_5.

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 5 & 4 \\ 0 & 1 & -3 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution

We write down the system of equations, ignoring the equation corresponding to the bottom row of the matrix since it is just $0 = 0$.

The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \quad + 2x_3 \quad + 5x_5 &= 4 \\ x_2 - 3x_3 \quad - \ \ x_5 &= 2 \\ x_4 + 3x_5 &= -7. \end{aligned}$$

The system does not immediately give us a solution, and so there is not a unique solution. Furthermore, there is no equation $0 = 1$ and so the system is not inconsistent. Therefore it must be a system with infinitely many solutions. We express each leading unknown, x_1, x_2 and x_4, in terms of the non-leading unknowns, x_3 and x_5.

This system is equivalent to

$$\begin{aligned} x_1 &= 4 - 2x_3 - 5x_5 \\ x_2 &= 2 + 3x_3 + x_5 \\ x_4 &= -7 - 3x_5. \end{aligned}$$

We can choose any values for x_3 and x_5, and the equations will give us the corresponding values of x_1, x_2 and x_4. So the system does have infinitely many solutions, one for each choice of values for x_3 and x_5. To obtain the general solution of the system, we set x_3 and x_5 equal to parameters.

Setting $x_3 = k$ and $x_5 = l$, $(k, l \in \mathbb{R})$, we obtain the general solution

$$\begin{aligned} x_1 &= 4 - 2k - 5l \\ x_2 &= 2 + 3k + l \\ x_3 &= k \\ x_4 &= -7 - 3l \\ x_5 &= l \quad (k, l \in \mathbb{R}). \end{aligned}$$

Exercise C11

Solve the system corresponding to each of the following row-reduced augmented matrices.

(a) Assume that the unknowns are x_1, x_2.

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{array}\right)$$

(b) Assume that the unknowns are x_1, x_2, x_3.

$$\left(\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

(c) Assume that the unknowns are x_1, x_2, x_3, x_4, x_5.

$$\left(\begin{array}{ccccc|r} 1 & 3 & 0 & 2 & 0 & -7 \\ 0 & 0 & 1 & -3 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(d) Assume that the unknowns are x_1, x_2, x_3, x_4.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

Row-reduction strategy

You have seen that, once the augmented matrix of a system of linear equations is reduced to row-reduced form, all the solutions of the system can easily be determined. You will now see that there is a systematic strategy for row-reducing *any* matrix using elementary row operations.

The idea of the strategy is to take each row of the matrix in turn as the current row, starting with the first. With row 1 as the current row, we carry out four steps, then with row 2 as the current row we carry out the same four steps, and so on. In outline the steps are as follows. In step 1 we identify the column for the current row's leading 1; in steps 2 and 3 we create a leading 1 in the current row. Finally in step 4 we make each entry above and below the leading 1 into a 0.

Strategy C1

To row-reduce a matrix using elementary row operations, carry out the following four steps, first with row 1 as the current row, then with row 2 as the current row, and so on, until

- **either** every row has been the current row,
- **or** step 1 is not possible.

1. Select the first column from the left that has at least one non-zero entry in or below the current row.
2. If the current row has a 0 in the selected column, interchange it with a row below that has a non-zero entry in that column.
3. If the entry in the current row and the selected column is c, multiply the current row by $1/c$ to create a leading 1.
4. Add suitable multiples of the current row to the other rows to make each entry above and below the leading 1 into a 0.

The strategy is illustrated in the following worked exercise, where the selected rows and columns of the matrix are highlighted by shading.

You do not need to include this level of detail in your solutions, or the shading, just the row operations and current matrix. It is always a good idea to include the check column to try to pick up any errors in arithmetic!

Worked Exercise C10

Use Strategy C1 to row-reduce the following matrix.

$$\begin{pmatrix} 2 & 4 & -2 & 2 & 4 \\ 3 & 6 & -3 & 6 & 5 \\ 2 & 1 & -11 & 2 & 6 \\ -1 & 1 & 10 & -7 & -2 \end{pmatrix}$$

Solution

Row 1 is the current row.

Step 1 identifies the column for the current row's leading 1: column 1. It is the first column from the left with at least one non-zero entry in, or below, the current row.

$$\begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \begin{pmatrix} 2 & 4 & -2 & 2 & 4 \\ 3 & 6 & -3 & 6 & 5 \\ 2 & 1 & -11 & 2 & 6 \\ -1 & 1 & 10 & -7 & -2 \end{pmatrix} \begin{array}{r} 10 \\ 17 \\ 0 \\ 1 \end{array}$$

Steps 2 and 3 create a leading 1 in the current row.

The current row does not have a 0 in the column selected: it has a 2, and so step 2 does not apply. In step 3 we multiply the current row by the reciprocal of 2; that is, by $\frac{1}{2}$.

In fact, when using this strategy to row-reduce a matrix there is often nothing to be done in step 2.

$$\mathbf{r}_1 \to \tfrac{1}{2}\mathbf{r}_1 \qquad \left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 5 \\ 3 & 6 & -3 & 6 & 5 & 17 \\ 2 & 1 & -11 & 2 & 6 & 0 \\ -1 & 1 & 10 & -7 & -2 & 1 \end{array}\right)$$

Step 4 makes each entry above and below the current leading 1 into a 0 by adding suitable multiples of the current row to the other rows.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 + \mathbf{r}_1 \end{array} \qquad \left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 3 & -1 & 2 \\ 0 & -3 & -9 & 0 & 2 & -10 \\ 0 & 3 & 9 & -6 & 0 & 6 \end{array}\right)$$

None of these row operations changes a row that is then used by another of these row operations, so they can be carried out in one go; in fact, this will always be the case for the row operations required in step 4.

Row 2 is the current row.

Step 1 identifies the column for the current row's leading 1: column 2.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \qquad \left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 3 & -1 & 2 \\ 0 & -3 & -9 & 0 & 2 & -10 \\ 0 & 3 & 9 & -6 & 0 & 6 \end{array}\right)$$

Steps 2 and 3 create a leading 1 in the current row.

The current row does have a 0 in the column selected, so in step 2 we interchange it with a row below that has a non-zero entry in that column. We choose to interchange it with row 4, although it does not matter which row of these we use.

$$\mathbf{r}_2 \leftrightarrow \mathbf{r}_4 \qquad \left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 5 \\ 0 & 3 & 9 & -6 & 0 & 6 \\ 0 & -3 & -9 & 0 & 2 & -10 \\ 0 & 0 & 0 & 3 & -1 & 2 \end{array}\right)$$

The current row has a 3 in the column selected, so in step 3 we multiply the current row by the reciprocal of 3; that is, $\frac{1}{3}$.

$$\mathbf{r}_2 \to \tfrac{1}{3}\mathbf{r}_2 \qquad \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 5 \\ 0 & 1 & 3 & -2 & 0 & 2 \\ 0 & -3 & -9 & 0 & 2 & -10 \\ 0 & 0 & 0 & 3 & -1 & 2 \end{array}\right)$$

Step 4 makes each entry above and below the current leading 1 into a 0 by adding suitable multiples of the current row to the other rows.

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 3\mathbf{r}_2 \\ \\ \end{array} \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -7 & 5 & 2 & 1 \\ 0 & 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & 0 & -6 & 2 & -4 \\ 0 & 0 & 0 & 3 & -1 & 2 \end{array}\right)$$

Row 3 is the current row.

Step 1 identifies the column for the current row's leading 1: column 4.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -7 & 5 & 2 & 1 \\ 0 & 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & 0 & -6 & 2 & -4 \\ 0 & 0 & 0 & 3 & -1 & 2 \end{array}\right)$$

Steps 2 and 3 create a leading 1 in the current row.

Here, step 2 does not apply, and in step 3 we multiply the current row by the reciprocal of -6; that is, $-\frac{1}{6}$.

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\tfrac{1}{6}\mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -7 & 5 & 2 & 1 \\ 0 & 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 3 & -1 & 2 \end{array}\right)$$

Step 4 makes each entry above and below the current leading 1 into a 0 by adding suitable multiples of the current row to the other rows.

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 5\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 - 3\mathbf{r}_3 \end{array} \qquad \left(\begin{array}{ccccc|c} 1 & 0 & -7 & 0 & \frac{11}{3} & -\frac{7}{3} \\ 0 & 1 & 3 & 0 & -\frac{2}{3} & \frac{10}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Row 4 now becomes the current row.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{array} \quad \left(\begin{array}{ccccc} 1 & 0 & -7 & 0 & \frac{11}{3} \\ 0 & 1 & 3 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{c} -\frac{7}{3} \\ \frac{10}{3} \\ \frac{2}{3} \\ 0 \end{array}$$

Here, step 1 is not possible so we finish. The matrix is in row-reduced form.

If the strategy has been carried out correctly, then the matrix will be in row-reduced form when we stop, as was the case in the worked exercise above.

In general, when applying the strategy we stop either after every row has been the current row and had the four steps carried out, or when we find that step 1 is not possible, which happens when there are one or more zero rows at the bottom of the matrix.

Exercise C12

Use Strategy C1 to row-reduce the following matrices.

(a) $$\begin{pmatrix} 1 & 5 & 1 & 4 & 5 & -1 \\ 1 & 5 & 3 & 12 & 11 & 3 \\ 3 & 15 & -1 & -4 & 3 & -6 \\ -2 & -10 & 1 & 2 & -7 & 6 \end{pmatrix}$$

(b) $$\begin{pmatrix} 0 & -8 & 8 & -14 \\ -1 & 0 & -4 & -6 \\ -1 & 8 & -12 & 8 \\ 2 & 8 & 0 & 24 \\ 1 & 4 & 0 & 14 \end{pmatrix}$$

Modifying the strategy

The strategy for row-reducing a matrix works for any matrix and can easily be programmed on a computer. But sometimes when carrying it out by hand we can spot places where carrying out a different row operation will make the calculations easier. Suppose we are working with the matrix below and that we have completed the four steps with row 1 as the current row. Row 2 becomes the current row and step 1 identifies column 2 for the current row's leading 1.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{cccc} 1 & 3 & 1 & 2 \\ 0 & 4 & 5 & 7 \\ 0 & 3 & 4 & 9 \end{array}\right) \quad \begin{array}{c} 7 \\ 16 \\ 16 \end{array}$$

Since the entry is not 0 there is nothing to be done in step 2. In step 3 we are now officially supposed to multiply the current row by $\frac{1}{4}$ in order to create a leading 1, but this will create inconvenient fractions as other entries in row 2. We can, however, spot a different row operation that will also create a leading 1 in the current row, but avoids creating fractions, namely $\mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3$, since subtracting the 3 from the 4 will create a leading 1. So we perform this alternative row operation as an unofficial version of step 3 and this gives us the matrix below.

$$\mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 \qquad \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 7 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 3 & 4 & 9 & 16 \end{array}\right)$$

We now carry out step 4 as normal:

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_2 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 0 & -2 & 8 & 7 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 15 & 16 \end{array}\right)$$

We then carry on with the third row as the current row.

In general, if you are trying to reduce a matrix to row-reduced form, you can use any elementary row operation. Note that even so it can sometimes be impossible to avoid fractions.

Until you are very familiar with row-reducing matrices, it is sensible to follow the systematic strategy very closely, considering modifications only at step 3.

When modifying the strategy and trying to identify an alternative row operation, it is important not to use rows above the current row, as the following exercise illustrates.

Exercise C13

Consider the following matrix where row 2 is the current row.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 7 \\ 0 & 4 & 5 & 7 & 16 \\ 0 & 3 & 4 & 9 & 16 \end{array}\right)$$

Carry out the following row operation and explain why it is not an appropriate alternative operation for step 3.

$$\mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1$$

When trying to choose an alternative row operation, rows below the current row can be used because the zeros at the beginning of these rows prevent them destroying the progress made so far.

Uniqueness

We have seen that there can be different ways to row-reduce a matrix. Whichever way you choose, you will always get the same answer. This is a consequence of the following theorem, which we state without proof.

Theorem C1

Every matrix has a unique row-reduced form.

Putting it all together

We now have all the techniques necessary for using Gauss–Jordan elimination to solve a system of linear equations using augmented matrices; we just need to put them all together as set out in the following strategy.

Strategy C2

To use Gauss–Jordan elimination to solve a given system of linear equations:

1. form the augmented matrix
2. row-reduce the augmented matrix to obtain its row-reduced form
3. solve the simplified system of linear equations.

Worked Exercise C11

Use Strategy C2 to solve the following system of linear equations.

$$\begin{aligned} 3x + 5y - 12z &= 4 \\ x + y \phantom{{}-12z} &= 2 \\ 2x + 3y - 4z &= 5 \end{aligned}$$

Solution

We first form the augmented matrix, label the rows and include a check column.

The augmented matrix is

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{ccc|c} 3 & 5 & -12 & 4 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & -4 & 5 \end{array}\right) \quad \begin{array}{c} 0 \\ 4 \\ 6 \end{array}$$

We now row-reduce this augmented matrix. However, to avoid creating awkward fractions, we perform the row operation $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ instead of $\mathbf{r}_1 \to \frac{1}{3}\mathbf{r}_1$.

$$\begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \\ \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 3 & 5 & -12 & 4 \\ 2 & 3 & -4 & 5 \end{array}\right) \quad \begin{array}{c} 4 \\ 0 \\ 6 \end{array}$$

We now carry on as usual, following the strategy.

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & -12 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right) \begin{array}{r} 4 \\ -12 \\ -2 \end{array}$$

$$\mathbf{r}_2 \to \tfrac{1}{2}\mathbf{r}_2 \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -6 & -1 \\ 0 & 1 & -4 & 1 \end{array}\right) \begin{array}{r} 4 \\ -6 \\ -2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right) \begin{array}{r} 10 \\ -6 \\ 4 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \tfrac{1}{2}\mathbf{r}_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 6 & 3 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{r} 10 \\ -6 \\ 2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 6\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 6\mathbf{r}_3 \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{r} -2 \\ 6 \\ 2 \end{array}$$

This matrix is in row-reduced form.

The corresponding system of equations is

$$\begin{aligned} x \qquad\quad &= -3, \\ y \qquad &= 5, \\ z &= 1. \end{aligned}$$

Thus the solution is $x = -3$, $y = 5$, $z = 1$.

Exercise C14

Use Strategy C2 to solve the following system of equations.

$$\begin{aligned} x_1 - 4x_2 - 4x_3 + 3x_4 + 6x_5 &= 2 \\ 2x_1 - 5x_2 - 6x_3 + 6x_4 + 9x_5 &= 3 \\ 2x_1 + 4x_2 \phantom{{}- 6x_3} + 9x_4 + 2x_5 &= 0 \end{aligned}$$

3 Matrix operations

In this section you will revise matrices and matrix operations such as matrix addition and matrix multiplication. You will also meet a useful operation called *transposition*.

3.1 Matrix arithmetic

Recall that a matrix with m rows and n columns is an $m \times n$ matrix. An $n \times n$ matrix is called a **square** matrix.

In general, we write $\mathbf{A}$ or (a_{ij}) to denote a matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}).$$

We call the entry in the ith row and jth column of a matrix $\mathbf{A}$ the (i,j)**-entry**, and often denote it by a_{ij}. Although matrices are usually distinguished in print by the use of bold typeface, when you handwrite them you do not need to underline them, unlike letters that represent vectors.

Matrices are closely related to vectors represented in component form. In Unit A1 you performed vector arithmetic on vectors in both $\mathbb{R}^2$ and $\mathbb{R}^3$, writing a vector in component form as a **row vector**. Such a row vector can be regarded, respectively, as a 1×2 matrix or a 1×3 matrix with real entries; the only difference is the lack of commas in the matrix representation. For example, consider the following 1×3 matrix and the corresponding row vector in $\mathbb{R}^3$:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad (1, 2, 3).$$

A **column vector** is a vector with the components written vertically; such a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be regarded as a matrix with real entries that has just a single column. For example, the following represents both a column vector in $\mathbb{R}^3$ and the corresponding 3×1 matrix:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

It should be clear from the context whether this object is a column vector, with a geometrical interpretation in $\mathbb{R}^3$, or a matrix with real entries.

A matrix may have any size, $m \times n$ for any natural numbers m and n, although we usually write a 1×1 matrix without the brackets and identify it with its single entry.

In this way, matrices can be regarded as a generalisation of vectors with *equality*, the *zero matrix* and the operations of *addition* and *scalar multiplication* defined similarly. Whereas for vectors we defined these in terms of the components, for matrices we define them in terms of the entries. The details are given in the box below.

Matrix arithmetic

Equality Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal** if all their corresponding entries agree. We write $\mathbf{A} = \mathbf{B}$.

Zero matrix The $m \times n$ **zero matrix** $\mathbf{0}_{m,n}$ is the $m \times n$ matrix in which all entries are 0. It is denoted by $\mathbf{0}$ when it is clear from the context which size is intended.

Addition The **sum** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is the $m \times n$ matrix $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$ obtained by adding the corresponding entries:

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}.$$

Addition of matrices of different sizes is not defined.

Negatives The **negative** of an $m \times n$ matrix $\mathbf{A} = (a_{ij})$ is the $m \times n$ matrix obtained by taking the negatives of its entries:

$$-\mathbf{A} = (-a_{ij}).$$

Subtraction The **difference** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is the $m \times n$ matrix $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})$ obtained by subtracting the corresponding entries:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{pmatrix}.$$

Subtraction of matrices of different sizes is not defined.

Scalar multiplication The **scalar multiple** of an $m \times n$ matrix $\mathbf{A} = (a_{ij})$ by a real number k is the $m \times n$ matrix obtained by multiplying each entry in turn by k.

$$k\mathbf{A} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix} = (ka_{ij}).$$

For example, consider the matrices below. The first pair are not equal because a pair of corresponding entries differ, and the second pair are not equal as they have different sizes:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The following are all examples of zero matrices:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The next worked exercise illustrates matrix addition. In the subsequent exercises you are asked to evaluate the results of various matrix operations.

Worked Exercise C12

Evaluate the following matrix sums, where possible.

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

Solution

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

(b) This sum is undefined since the matrices are of different sizes.

Exercise C15

Evaluate the following matrix sums, where possible.

(a) $\begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & 4 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 6 & -2 \\ 1 & 8 & 2 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 9 \\ 1 & 0 & 4 \\ 3 & -4 & 1 \end{pmatrix}$

Exercise C16

Evaluate the following matrix differences, where possible.

(a) $\begin{pmatrix} 3 & 0 \\ 2 & 7 \end{pmatrix} - \begin{pmatrix} 10 & 3 \\ 1 & 5 \\ 15 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 8 & 12 \\ 7 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 10 & 2 \\ 4 & 9 & 21 \end{pmatrix}$

Exercise C17

Let

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 2 & 3 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ -2 & -7 \\ 3 & 5 \end{pmatrix}.$$

Evaluate the following.

(a) $4\mathbf{A}$ (b) $4\mathbf{B}$ (c) $4\mathbf{A} + 4\mathbf{B}$ (d) $4(\mathbf{A} + \mathbf{B})$

In Exercise C15 you should have found that parts (a) and (b) gave the same answer; this is because matrix addition is *commutative*. In fact, matrix addition has the same properties as the additive properties of the real numbers, $\mathbb{R}$, given in Unit A2 *Number systems*. Before listing these properties, we need the following notation:

$M_{m,n}$ denotes the set of all $m \times n$ matrices with real entries.

We can now talk about *arithmetic in* $M_{m,n}$ and the properties it satisfies.

Addition in $M_{m,n}$

A1 Closure For all $\mathbf{A}, \mathbf{B} \in M_{m,n}$,

$$\mathbf{A} + \mathbf{B} \in M_{m,n}.$$

A2 Associativity For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_{m,n}$,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

A3 Additive identity For all $\mathbf{A} \in M_{m,n}$ and $\mathbf{0} \in M_{m,n}$,

$$\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}.$$

A4 Additive inverses For each $\mathbf{A} \in M_{m,n}$, there is a matrix $-\mathbf{A} \in M_{m,n}$ such that

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0} = -\mathbf{A} + \mathbf{A}.$$

A5 Commutativity For all $\mathbf{A}, \mathbf{B} \in M_{m,n}$,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

The matrix $\mathbf{0}$ is known as the **additive identity** in $M_{m,n}$, and the matrix $-\mathbf{A}$ in property A4 is known as the **additive inverse** of $\mathbf{A}$.

These properties follow from the definition of matrix addition and the corresponding properties of the reals. The next worked exercise proves the closure property (A1) and the commutative property (A5); you are asked to prove the remaining properties in the following exercise.

Worked Exercise C13

By using the corresponding properties for the reals, prove that the following properties hold for $M_{m,n}$ under addition.

(a) The closure property (A1): $\mathbf{A} + \mathbf{B} \in M_{m,n}$.

(b) The commutative property (A5): $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Solution

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$.

(a) To prove closure, we add the two matrices and check that the result is another $m \times n$ matrix with real entries.

From the definition of matrix addition, the sum $\mathbf{A} + \mathbf{B}$ is the $m \times n$ matrix with entries $(a_{ij} + b_{ij})$ obtained by adding the corresponding entries.

Since a_{ij} and b_{ij} are real numbers, $a_{ij} + b_{ij} \in \mathbb{R}$. Hence $\mathbf{A} + \mathbf{B} \in M_{m,n}$.

(b) To prove the commutative property, we add the corresponding entries of these two matrices.

The (i, j)-entry of the matrix $\mathbf{A} + \mathbf{B}$ is $a_{ij} + b_{ij}$, and that of $\mathbf{B} + \mathbf{A}$ is $b_{ij} + a_{ij}$.

Since a_{ij} and b_{ij} are real numbers, $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. Thus

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = \mathbf{B} + \mathbf{A}.$$

Exercise C18

By using the corresponding properties for the reals, prove that the following properties hold for $M_{m,n}$ under addition.

(a) The associative property (A2): $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

(b) The identity property (A3): $\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}$.

(c) The inverses property (A4): $\mathbf{A} + (-\mathbf{A}) = \mathbf{0} = -\mathbf{A} + \mathbf{A}$.

Recall from Subsection 3.1 of Unit B1 *Symmetry and groups* that a set with a binary operation is a **group** if the following four axioms hold:

G1 (closure), G2 (associativity), G3 (identity) and G4 (inverses).

The first four properties (A1–A4) of matrix addition show that the set of all $m \times n$ matrices with real entries satisfies these four properties; that is, $(M_{m,n}, +)$ is a group with additive identity the zero matrix $\mathbf{0}$, and $-\mathbf{A}$ the additive inverse of $\mathbf{A}$. The final property (A5) shows that it is in fact an abelian group.

Although this unit concentrates on $M_{m,n}$, the set of matrices with real entries, some other sets of $m \times n$ matrices also form a group under addition. For example, the set of $m \times n$ matrices with entries in $\mathbb{Z}$, and those with entries in $\mathbb{C}$, both form a group under addition. However, the set of $m \times n$ matrices with entries in $\mathbb{N}$ does not form a group under addition, since this set of matrices contains neither the zero matrix $\mathbf{0}$, nor the additive inverse $-\mathbf{A}$ of a matrix $\mathbf{A}$ in the set.

Finally in this subsection we return to scalar multiplication of matrices. Recall, from Unit A2, that the reals satisfy a *distributive property* (D1) combining addition and multiplication:

$$a \times (b + c) = (a \times b) + (a \times c), \quad \text{for all } a, b, c \in \mathbb{R}.$$

It turns out that the corresponding property holds for addition and scalar multiplication of matrices; you saw one example of this in Exercise C17(c) and (d) where $4(\mathbf{A} + \mathbf{B})$ and $4\mathbf{A} + 4\mathbf{B}$ were equal.

Combining addition and scalar multiplication in $M_{m,n}$

D1 Distributivity For all $\mathbf{A}, \mathbf{B} \in M_{m,n}$ and $k \in \mathbb{R}$,

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

You are asked to prove that this property holds in the next exercise.

Exercise C19

By using the corresponding property for the reals, prove that the distributive property (D1) holds for $M_{m,n}$:

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

3.2 Matrix multiplication

In the previous subsection you saw that matrix addition and scalar multiplication can be defined in terms of matrix entries, much like the corresponding operations for vectors. In this subsection you will revise *matrix multiplication*, which can also be defined in terms of matrix entries, much like the corresponding operation for vectors – the *scalar product*.

Recall from Unit A1 that the scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Matrix multiplication is a generalisation of this idea.

To form the product of two matrices $\mathbf{A}$ and $\mathbf{B}$, we combine the rows of $\mathbf{A}$ with the columns of $\mathbf{B}$. The (i, j)-entry of the product $\mathbf{AB}$ is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{B}$ and adding the results.

This product is only possible if the number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$.

For example, let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

The number of columns of $\mathbf{A}$ is equal to the number of rows of $\mathbf{B}$, so it is possible to find the product $\mathbf{AB}$.

To obtain the $(1,1)$-entry of the product $\mathbf{AB}$ we combine the *first* row of $\mathbf{A}$ with the *first* column of $\mathbf{B}$:

$$(1 \times 1) + (2 \times 4) + (3 \times 7) = 1 + 8 + 21 = 30.$$

Next, to obtain the $(1,2)$-entry of the product $\mathbf{AB}$, we combine the *first* row of $\mathbf{A}$ with the *second* column of $\mathbf{B}$:

$$(1 \times 2) + (2 \times 5) + (3 \times 8) = 2 + 10 + 24 = 36.$$

Then to obtain the $(1,3)$-entry of the product $\mathbf{AB}$, we combine the *first* row of $\mathbf{A}$ with the *third* column of $\mathbf{B}$:

$$(1 \times 3) + (2 \times 6) + (3 \times 9) = 3 + 12 + 27 = 42.$$

To obtain the entries in the *second* row of the product $\mathbf{AB}$, we combine the *second* row of $\mathbf{A}$ with each of the columns of $\mathbf{B}$ in turn.

In the end we obtain 2×3 entries in the product $\mathbf{AB}$; this matrix has 2 rows and 3 columns as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \end{pmatrix}$$

One way to remember how to multiply matrices $\mathbf{A}$ and $\mathbf{B}$ is to picture running along the rows of $\mathbf{A}$ and then diving down the columns of $\mathbf{B}$. The example pictured in Figure 13 gives the $(1,2)$-entry.

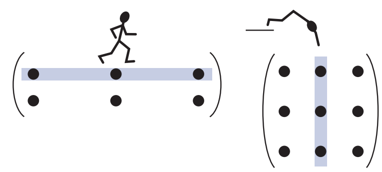

Figure 13 Running along and diving in

Definition

The **product** of an $m \times n$ matrix $\mathbf{A}$ with an $n \times p$ matrix $\mathbf{B}$ is the $m \times p$ matrix $\mathbf{AB}$ whose (i,j)-entry is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{B}$ and adding the results.

In symbols, if $\mathbf{C} = \mathbf{AB}$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

The product $\mathbf{AB}$ is not defined when the number of columns of the matrix $\mathbf{A}$ is not equal to the number of rows of the matrix $\mathbf{B}$.

Schematically, this can be shown as follows.

$$
\underset{\text{row } i}{m\updownarrow\overset{\longleftarrow n \longrightarrow}{\begin{pmatrix} & & \\ * & \cdots & * \\ & & \end{pmatrix}}}
\quad
\underset{\text{column } j}{n\updownarrow\overset{\longleftarrow p \longrightarrow}{\begin{pmatrix} & * & \\ & \vdots & \\ & * & \end{pmatrix}}}
=
\underset{(i,j)\text{-entry}}{m\updownarrow\overset{\longleftarrow p \longrightarrow}{\begin{pmatrix} & & \\ & * & \\ & & \end{pmatrix}}}
$$

Worked Exercise C14

Evaluate (where possible) the matrix products **AB**, where:

(a) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & 4 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -2 \end{pmatrix}$.

Solution

(a) The matrix **A** has 2 columns and the matrix **B** has 2 rows, so the product **AB** can be formed.

The product of a 2×2 matrix with a 2×3 one is a 2×3 matrix.

When evaluating a product of matrices, it is advisable to find the entries systematically, either row by row, or column by column. Here, we find the entries row by row.

To find the $(1,1)$-entry of **AB**, we multiply each entry in the *first* row of **A** by the corresponding entry in the *first* column of **B**:

$$(2 \times 3) + (1 \times 1) = 7.$$

Next, to find the $(1,2)$-entry of **AB**, we apply the same procedure to the *first* row of **A** and the *second* column of **B**:

$$(2 \times (-2)) + (1 \times 1) = -3.$$

Next, to find the $(1,3)$-entry of **AB**, we apply the same procedure to the *first* row of **A** and the *third* column of **B**:

$$(2 \times 0) + (1 \times 4) = 4.$$

Together, these give the first row of **AB**:

$$\begin{pmatrix} 7 & -3 & 4 \\ * & * & * \end{pmatrix}$$

We continue by finding the $(2,1)$-entry of **AB** then the $(2,2)$-entry and finally the $(2,3)$-entry, by applying the same procedure to the *second* row of **A** with the columns of **B** in turn. This gives the second row of the product **AB**:

$$\begin{pmatrix} 7 & -3 & 4 \\ -9 & 6 & 0 \end{pmatrix}$$

We have

$$\begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -3 & 4 \\ -9 & 6 & 0 \end{pmatrix}.$$

(b) This product is not defined, because the matrix $\mathbf{A}$ has 2 columns and the matrix $\mathbf{B}$ has 1 row.

Exercise C20

Evaluate the following matrix products, where possible.

(a) $\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -4 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & 3 & 0 \\ 4 & 1 & -1 \end{pmatrix}$

In the previous subsection you saw that addition on the set $M_{m,n}$ of $m \times n$ matrices satisfies the usual properties (A1–A5) for addition. For multiplication of matrices things are not so straightforward. To start with, if $m \neq n$ then the product of two matrices in the set $M_{m,n}$ is not even defined.

So when we consider properties of matrix multiplication we are only interested in products that can be formed. For example, we can say that matrix multiplication is associative because, *whenever these products can be formed*, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. You will prove this result in Unit C3.

In the next exercise you are asked to prove that matrix multiplication is not commutative.

Exercise C21

(a) Prove that the products $\mathbf{AB}$ and $\mathbf{BA}$ are the same size if and only if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size.

(b) Prove that matrix multiplication of square matrices of the same size is not commutative by giving a counterexample; that is, find two 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB} \neq \mathbf{BA}$.

The fact that matrix multiplication is not commutative means that it is important to describe a matrix product carefully. We say that $\mathbf{AB}$ is the matrix $\mathbf{A}$ *multiplied on the right* by the matrix $\mathbf{B}$, or the matrix $\mathbf{B}$ *multiplied on the left* by the matrix $\mathbf{A}$.

You have seen that the distributive property (D1) holds for multiplication of a matrix by a scalar. Matrix multiplication is also distributive because, *whenever these products can be formed*, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$. The proof of this is not hard, but it is not very illuminating, so is not given here.

Diagonal and triangular matrices

The entries of a square matrix from the top left-hand corner to the bottom right-hand corner are the **diagonal** entries; the diagonal entries form the **main diagonal** of the matrix. In some texts the main diagonal is called the *leading* or *principal diagonal*. For a square matrix $\mathbf{A} = (a_{ij})$ of size $n \times n$, the diagonal entries are

$$a_{11},\ a_{22},\ \ldots,\ a_{nn}.$$

A matrix that has its only non-zero entries on the main diagonal can be useful.

Definition

A **diagonal matrix** is a square matrix each of whose non-diagonal entries is zero.

For example, the following are diagonal matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To see how diagonal matrices multiply, try the following exercise.

Exercise C22

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}.$$

Evaluate the following products.

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $\mathbf{ABC}$

The product of two diagonal matrices is another diagonal matrix, and the ith diagonal entry of the product is the product of the ith diagonal entries of the matrices being multiplied. Multiplication of *diagonal matrices* is therefore commutative.

More generally, positive **powers** of square matrices are defined as expected:

$$\mathbf{A}^2 = \mathbf{AA}, \quad \mathbf{A}^3 = \mathbf{AAA}, \quad \ldots.$$

Therefore, finding powers of diagonal matrices is straightforward and you will see how this fact can be used to find powers of other square matrices in Unit C4 *Eigenvectors*.

A square matrix with each entry *below* the main diagonal equal to zero is called an **upper triangular matrix**. Similarly, a square matrix with each entry *above* the main diagonal equal to zero is called a **lower triangular matrix**. A square row-reduced matrix is an upper triangular matrix. A square matrix that is both upper triangular and lower triangular is necessarily a diagonal matrix.

Exercise C23

State which of the following matrices are diagonal, upper triangular or lower triangular.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

Identity matrix

You have seen that there are matrices corresponding to the number 0, which is the additive identity in the reals. These matrices are the zero matrices $\mathbf{0}_{m,n}$, each of which is the additive identity in $M_{m,n}$. There are also matrices corresponding to the number 1, which is the multiplicative identity in the reals. These matrices are square matrices called the *identity matrices*, denoted by $\mathbf{I}_n$. The subscript n indicates that the matrix is an $n \times n$ matrix; however, as with the zero matrix, the identity matrix is written simply as $\mathbf{I}$ when the size is clear from the context.

Definition

The **identity matrix** $\mathbf{I}_n$ is the $n \times n$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Each of the entries is 0 except those on the main diagonal, which are all 1.

For example, the identity matrices $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we multiply a 3×2 matrix on the left by $\mathbf{I}_3$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

If we multiply the same 3×2 matrix on the right by $\mathbf{I}_2$ we obtain

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}.$$

Here, a, b, c, d, e and f are any real numbers. In both cases, the matrix is unchanged.

Theorem C2

Let $\mathbf{A}$ be an $m \times n$ matrix. Then

$$\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n.$$

You are asked to prove this theorem in the next exercise.

Exercise C24

Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix. Prove Theorem C2; that is, prove that $\mathbf{I}_m\mathbf{A} = \mathbf{A}$ and $\mathbf{A}\mathbf{I}_n = \mathbf{A}$.

Hint: Notice that the entries in the ith row of $\mathbf{I}_m$ are all 0 except the entry in the ith position, which is 1.

3.3 Transposition of matrices

There is a simple operation that we can perform on matrices. This operation, called *transposition* or *taking the transpose*, entails interchanging the rows with the columns of the matrix. Thus the transpose of the matrix $\mathbf{A}$, denoted by $\mathbf{A}^T$, has the rows of $\mathbf{A}$ as its columns, taken in the same order. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 7 \\ -6 & 1 \\ 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & -6 & 0 \\ 7 & 1 & 4 \end{pmatrix}.$$

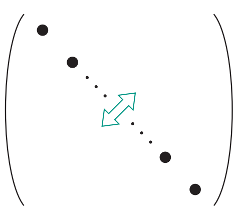

Figure 14 Transposition as reflecting in the main diagonal

Transposition of a *square* matrix can be thought of as reflecting the matrix in the main diagonal, as illustrated in Figure 14.

Definition

The **transpose** of an $m \times n$ matrix $\mathbf{A}$ is the $n \times m$ matrix $\mathbf{A}^T$ whose (i, j)-entry is the (j, i)-entry of $\mathbf{A}$.

Exercise C25

Write down the transpose of each of the following matrices.

(a) $\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -6 & 10 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -5 \\ 4 & 7 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 10 & 4 & 6 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

The identity matrix $\mathbf{I}$ is not changed by taking the transpose; that is, $\mathbf{I}^T = \mathbf{I}$. In fact, $\mathbf{A}^T = \mathbf{A}$ for all *diagonal* matrices; you saw one such example in Exercise C25(d).

The operation of transposition has some other useful properties as you will now see.

The rows of the matrix $\mathbf{A}$ form the columns of the matrix $\mathbf{A}^T$, and the columns of $\mathbf{A}^T$ form the rows of $(\mathbf{A}^T)^T$. Therefore the rows of $\mathbf{A}$ form the rows of $(\mathbf{A}^T)^T$; that is, these two matrices are equal:

$$(\mathbf{A}^T)^T = \mathbf{A}.$$

Exercise C26

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(a) Find $\mathbf{A}^T$, $\mathbf{B}^T$ and $(\mathbf{A}+\mathbf{B})^T$, and verify that $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(b) Find $\mathbf{C}^T$ and $(\mathbf{AC})^T$, and find an equation relating $(\mathbf{AC})^T$, $\mathbf{A}^T$ and $\mathbf{C}^T$.

The relationships satisfied by the matrices in Exercise C26 hold in general.

Properties of matrix transposition

Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. Then:

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times p$ matrix. Then

3. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Symmetric matrices

Some square matrices remain unchanged when transposed. These matrices are called *symmetric* matrices, since they are symmetrical about the main diagonal.

> **Definition**
>
> A square matrix $\mathbf{A}$ is **symmetric** if $\mathbf{A}^T = \mathbf{A}$.

Since $\mathbf{A}^T = \mathbf{A}$ for all diagonal matrices, all diagonal matrices are symmetric. Here are other examples of symmetric matrices:

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}, \quad \begin{pmatrix} -5 & 2 \\ 2 & 3 \end{pmatrix}.$$

3.4 Matrix form of a system of linear equations

In this subsection you will see how a system of linear equations can be expressed in *matrix form* as a product of matrices. This contrasts with the augmented matrices you met in Subsection 2.1, which are an abbreviated notation for the system and involve no products of matrices.

Consider the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 6 \\ x_2 + x_3 &= 1 \\ x_1 + 3x_2 + 5x_3 &= 10 \end{aligned}$$

We can write this system as a matrix equation:

$$\begin{pmatrix} x_1 + 2x_2 + 4x_3 \\ x_2 + x_3 \\ x_1 + 3x_2 + 5x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}.$$

Now the 3×1 matrix on the left can be expressed as the product of two matrices, namely the 3×3 matrix of the coefficients and the 3×1 matrix of the unknowns:

$$\begin{pmatrix} x_1 + 2x_2 + 4x_3 \\ x_2 + x_3 \\ x_1 + 3x_2 + 5x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus we have the matrix equation

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}.$$

Similarly, we can express any system of linear equations

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1, \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2, \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

as a matrix product. Let the matrix of coefficients be $\mathbf{A}$, the **coefficient matrix** of the system, that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Let the matrix of unknowns be $\mathbf{x}$, and let the matrix of constant terms be $\mathbf{b}$, so

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The system can then be expressed in **matrix form** as

$$\mathbf{Ax} = \mathbf{b},$$

or in full as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Writing a system of linear equations in matrix form will allow us to manipulate the system using matrix multiplication.

4 Matrix inverses

In this section you will investigate the multiplicative properties of *square* matrices and the existence of multiplicative inverses.

4.1 Matrix inverses

In Section 3 you saw that matrix addition in $M_{m,n}$ satisfies the usual properties (A1–A5) for addition, but things are not so straightforward for multiplication of matrices.

If we restrict our attention to the set $M_{n,n}$ of *square* matrices with real entries, then products of these matrices can always be formed, and so the following properties hold.

Multiplication in $M_{n,n}$

M1 Closure For all $\mathbf{A}, \mathbf{B} \in M_{n,n}$,

$$\mathbf{AB} \in M_{n,n}.$$

M2 Associativity For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_{n,n}$,

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}).$$

M3 Multiplicative identity For all $\mathbf{A} \in M_{n,n}$,

$$\mathbf{AI}_n = \mathbf{A} = \mathbf{I}_n\mathbf{A}.$$

The closure property (M1) follows from the definition of matrix multiplication and the associative property (M2) will be proved in Unit C3. The multiplicative identity property (M3) holds by Theorem C2 and we say that $\mathbf{I}_n$ is the **multiplicative identity** in $M_{n,n}$.

You saw that matrix multiplication is not commutative, even for square matrices, and so the commutative property (M5) does not hold for matrix multiplication in $M_{n,n}$. The distributive property (D1) does hold for matrix addition and matrix multiplication in $M_{n,n}$; that is, matrix multiplication is *distributive* over matrix addition. However, because matrix multiplication is not commutative we have to consider multiplying on the right and left separately.

Combining addition and multiplication in $M_{n,n}$

D1 Distributivity For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in M_{n,n}$,

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC},$$

and

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}.$$

You may have noticed that one other property is missing from the list of multiplicative properties, namely the multiplicative inverses property (M4).

Recall, from Exercise C21(a), that the products $\mathbf{AB}$ and $\mathbf{BA}$ are the same size if and only if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size.

We say that $\mathbf{B}$ is a **multiplicative inverse** of $\mathbf{A}$ in $M_{n,n}$ if $\mathbf{A}, \mathbf{B} \in M_{n,n}$ and $\mathbf{AB} = \mathbf{I}_n = \mathbf{BA}$. In fact, because the additive inverse of a matrix is usually called the *negative* of the matrix, the multiplicative inverse is usually called the *inverse* of a matrix, where the context is clear.

We now investigate the existence of multiplicative inverses.

Many square matrices do have multiplicative inverses, for example,

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \text{ is an inverse of } \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \text{ is an inverse of } \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}$$

since

$$\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix} \begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Just as a real number has at most one multiplicative inverse, or reciprocal, a square matrix has at most one inverse, as we now prove.

Theorem C3

If a square matrix has an inverse, then this inverse is unique.

Proof Let $\mathbf{A}$ be a square matrix, and suppose that $\mathbf{B}$ and $\mathbf{C}$ are both inverses of $\mathbf{A}$. Then $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$ and $\mathbf{AC} = \mathbf{I} = \mathbf{CA}$.

We consider the product $\mathbf{CAB} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B}$.

Multiplying the equation $\mathbf{AB} = \mathbf{I}$ on the left by $\mathbf{C}$, we have

$$\mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C},$$

while multiplying the equation $\mathbf{CA} = \mathbf{I}$ on the right by $\mathbf{B}$ gives

$$(\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}.$$

Since matrix multiplication is associative, it follows that $\mathbf{B} = \mathbf{C}$. ■

Certainly a square zero matrix has no inverse (just as the real number 0 has no reciprocal), since if $\mathbf{0}$ is a square zero matrix, then any product of $\mathbf{0}$ and another matrix is a zero matrix, and so there is no matrix $\mathbf{B}$ such that $\mathbf{0B} = \mathbf{I}$. However, it is natural to ask whether or not every *non-zero* square matrix has an inverse. The next exercise demonstrates that the answer to this question is *no*: it gives an example of a non-zero square matrix with no inverse.

Exercise C27

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Prove that there is no matrix $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathbf{AB} = \mathbf{I}$.

In fact, there are many non-zero square matrices with no inverse. The next theorem gives an infinite class of such matrices.

Theorem C4

A square matrix with a zero row has no inverse.

Proof Let $\mathbf{A}$ be a square matrix, one of whose rows, say row i, is a zero row. Then if $\mathbf{B}$ is any matrix of the same size as $\mathbf{A}$, the (i,i)-entry of $\mathbf{AB}$ is 0, since it is obtained by multiplying each entry in row i of $\mathbf{A}$ (a zero row) by the corresponding entry in column i of $\mathbf{B}$. But the (i,i)-entry of $\mathbf{I}$ is 1, which shows that there is no matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$. Hence $\mathbf{A}$ has no inverse. ■

Definition

A square matrix that has an inverse is called **invertible**.

An invertible matrix is necessarily a square matrix in order for the products $\mathbf{AB}$ and $\mathbf{BA}$ to exist and be equal.

Since we know by Theorem C3 that if a matrix has an inverse, then this inverse is unique, we denote the unique inverse of an invertible matrix $\mathbf{A}$ by $\mathbf{A}^{-1}$. Thus, for any invertible matrix $\mathbf{A}$,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}.$$

Both $\mathbf{A}$ and $\mathbf{A}^{-1}$ are square matrices of the same size. It follows from these equations that if $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is also invertible, with inverse $\mathbf{A}$; that is,

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

In other words, the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ are *inverses of each other*.

The next worked exercise and the following exercises give some other useful facts about inverses of matrices.

Worked Exercise C15

Let $\mathbf{A}$ be an invertible matrix. Prove that $\mathbf{A}^T$ is invertible, and that $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Solution

The *transpose* of a matrix is the matrix with the rows and columns interchanged.

To prove that $\mathbf{A}^T$ is invertible, with inverse $(\mathbf{A}^{-1})^T$, we have to show that

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = \mathbf{I} = (\mathbf{A}^{-1})^T\mathbf{A}^T.$$

Recall that $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$.

We have

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}^T = \mathbf{I},$$

and, similarly,

$$(\mathbf{A}^{-1})^T\mathbf{A}^T = (\mathbf{A}\mathbf{A}^{-1})^T = \mathbf{I}^T = \mathbf{I}.$$

Therefore $\mathbf{A}^T$ is invertible with inverse $(\mathbf{A}^{-1})^T$.

Exercise C28

Prove that the identity matrix $\mathbf{I}$ is invertible, and that $\mathbf{I}^{-1} = \mathbf{I}$.

Exercise C29

Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices of the same size. Prove that $\mathbf{A}\mathbf{B}$ is invertible, and that $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Notice the reversal of the order of the matrices in the identity

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

This result of Exercise C29 extends to products of any number of matrices; it can be proved using this result and mathematical induction.

Theorem C5

Let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_k$ be invertible matrices of the same size. Then the product $\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k$ is invertible, with

$$(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1}\cdots\mathbf{A}_1^{-1}.$$

You saw in Subsection 3.1 that $(M_{m,n}, +)$, the set of all $m \times n$ matrices with real entries, forms a group under addition. The results of Exercises C28 and C29, together with the properties M1–M3 for matrix multiplication in $M_{n,n}$, can be used to show that the set of all *invertible* matrices of a particular size and with real entries forms a group under *matrix multiplication*. The restriction of the set to include only invertible matrices is important: without this, the axiom G4 (inverses) clearly fails since, for example, the zero matrix has no inverse.

Theorem C6

The set of all invertible $n \times n$ matrices with real entries forms a group under matrix multiplication.

Proof We check the four group axioms.

G1 Closure Exercise C29 showed that if $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices then their product $\mathbf{AB}$ is invertible. The product $\mathbf{AB}$ is an $n \times n$ matrix, so group axiom G1 (closure) holds for this set.

G2 Associativity The associative property (M2) holds for matrix multiplication in $M_{n,n}$, so group axiom G2 (associativity) holds.

G3 Identity The identity property (M3) holds for matrix multiplication in $M_{n,n}$, and Exercise C28 shows that $\mathbf{I}_n$ is in the set of all invertible $n \times n$ matrices with real entries. Therefore group axiom G3 (identity) holds.

G4 Inverses The set of all invertible $n \times n$ matrices with real entries is a subset of $M_{n,n}$. By definition every matrix in the set of invertible matrices has an inverse, and this inverse is itself invertible and therefore in the set, so axiom G4 (inverses) holds.

Hence the set of all invertible $n \times n$ matrices with real entries under the operation of matrix multiplication satisfies the four group axioms, and so is a group. ■

4.2 Invertibility Theorem

The following two questions may already have occurred to you as you worked through the previous subsection. First, how can we determine whether or not a given square matrix is invertible? Second, if we know that a matrix is invertible, how can we find its inverse? The next theorem, which we will prove in Subsection 4.5, answers both these questions.

Theorem C7 Invertibility Theorem

(a) A square matrix is invertible if and only if its row-reduced form is $\mathbf{I}$.

(b) Any sequence of elementary row operations that transforms a matrix $\mathbf{A}$ to $\mathbf{I}$ also transforms $\mathbf{I}$ to $\mathbf{A}^{-1}$.

To illustrate this theorem, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}.$$

Suppose that we wish to determine whether or not $\mathbf{A}$ is invertible and, if it is, to find $\mathbf{A}^{-1}$.

Below, on the left, we row-reduce $\mathbf{A}$ in the usual way. On the right, we perform the same sequence of elementary row operations on the 2×2 identity matrix.

$$\begin{array}{llll}
\begin{matrix}\mathbf{r}_1 \\ \mathbf{r}_2\end{matrix} & \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix} & \begin{matrix}\mathbf{r}_1 \\ \mathbf{r}_2\end{matrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix} & \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\
\mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 & \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} & \mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 & \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \\
\mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & \begin{pmatrix} 3 & -1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}
\end{array}$$

The row-reduced form of $\mathbf{A}$ is $\mathbf{I}$, and so we conclude from the first part of the Invertibility Theorem that $\mathbf{A}$ is an invertible matrix.

By the second part of the Invertibility Theorem, the final matrix on the right above must be $\mathbf{A}^{-1}$; that is,

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -1 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

You should check that this matrix is indeed the inverse of $\mathbf{A}$.

To apply the Invertibility Theorem to find the inverse of a matrix $\mathbf{A}$, we have to perform the same sequence of elementary row operations on both $\mathbf{A}$ and $\mathbf{I}$. We can do this conveniently in the following way. We begin by writing $\mathbf{A}$ and $\mathbf{I}$ alongside each other, separated by a vertical line, giving a larger matrix, which we may denote by $(\mathbf{A} \mid \mathbf{I})$. We then row-reduce $(\mathbf{A} \mid \mathbf{I})$ in the usual way (with the check column included). When we do this, the above calculation is as follows.

$$\begin{array}{ll}
\begin{array}{l}\mathbf{r}_1\\ \mathbf{r}_2\end{array} & \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0\\ 2 & 9 & 0 & 1\end{array}\right) \begin{array}{c}5\\ 12\end{array}\\[2ex]
\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0\\ 0 & 3 & -2 & 1\end{array}\right) \begin{array}{c}5\\ 2\end{array}\\[2ex]
\mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 & \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0\\ 0 & 1 & -\frac{2}{3} & \frac{1}{3}\end{array}\right) \begin{array}{c}5\\ \frac{2}{3}\end{array}\\[2ex]
\mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & \left(\begin{array}{cc|cc} 1 & 0 & 3 & -1\\ 0 & 1 & -\frac{2}{3} & \frac{1}{3}\end{array}\right) \begin{array}{c}3\\ \frac{2}{3}\end{array}
\end{array}$$

Thus the Invertibility Theorem (Theorem C7) gives us the following strategy.

Strategy C3

To determine whether or not a given square matrix $\mathbf{A}$ is invertible, and to find its inverse if it is, do the following.

Write down $(\mathbf{A} \mid \mathbf{I})$, and row-reduce it until the left half is in row-reduced form.

- If the left half is the identity matrix, then the right half is $\mathbf{A}^{-1}$.
- Otherwise, $\mathbf{A}$ is not invertible.

You may find it helpful to remember the following scheme for this strategy:

$$\begin{array}{c}(\mathbf{A} \mid \mathbf{I})\\ \downarrow\\ (\mathbf{I} \mid \mathbf{A}^{-1}).\end{array}$$

Strategy C3 is most useful for matrices of size 3×3 and larger. In Section 5 you will revise a quick method for determining whether or not a 2×2 matrix is invertible, and for writing down its inverse if it is invertible.

Worked Exercise C16

Determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2\\ -1 & 0 & -4\\ 3 & 2 & 10\end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5\\ 3 & 1 & 7\\ 2 & 4 & 8\end{pmatrix}$

Solution

We use Strategy C3, and again add the row-sum check to help pick up any arithmetical errors.

(a) We row-reduce the matrix $(\mathbf{A} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & -4 & 0 & 1 & 0 \\ 3 & 2 & 10 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{r} 5 \\ -4 \\ 16 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_1 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -1 & 4 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{r} 5 \\ 1 \\ 1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right) \quad \begin{array}{r} 4 \\ 1 \\ 2 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \frac{1}{2}\mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \quad \begin{array}{r} 4 \\ 1 \\ 1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 4\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \\ \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 4 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right) \quad \begin{array}{r} 0 \\ 3 \\ 1 \end{array}$$

The left half has been reduced to $\mathbf{I}$, so $\mathbf{A}$ is invertible; its inverse is given by the right half, that is,

$$\mathbf{A}^{-1} = \begin{pmatrix} 4 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(b) We row-reduce $(\mathbf{B} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 3 & 5 & 1 & 0 & 0 \\ 3 & 1 & 7 & 0 & 1 & 0 \\ 2 & 4 & 8 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{r} 10 \\ 12 \\ 15 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -8 & -8 & -3 & 1 & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{r} 10 \\ -18 \\ -5 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{8}\mathbf{r}_2 \\ \\ \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{r} 10 \\ \frac{9}{4} \\ -5 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2 \end{array} \qquad \left(\begin{array}{rrr|rrr} 1 & 0 & 2 & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 1 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{5}{4} & -\frac{1}{4} & 1 \end{array}\right) \quad \begin{array}{r} \frac{13}{4} \\ \frac{9}{4} \\ -\frac{1}{2} \end{array}$$

The left half is now in row-reduced form, but is not the identity matrix. Therefore $\mathbf{B}$ is not invertible.

If, for a general matrix $\mathbf{A}$, it becomes clear while you are row-reducing $(\mathbf{A} \mid \mathbf{I})$ that the left half will not reduce to the identity matrix (for example, if a zero row appears in the left half), then you can stop the row-reduction immediately, and conclude that $\mathbf{A}$ is not invertible. There is no point in continuing until the left half is in row-reduced form.

Exercise C30

Use Strategy C3 to determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 1 & -4 \\ 2 & 1 & -6 \\ -3 & -1 & 9 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 4 \\ 5 & 10 & 5 \end{pmatrix}$

4.3 Invertibility and systems of linear equations

We can use matrix inverses to give us another method for solving certain systems of linear equations.

Consider the system that we solved by Gauss–Jordan elimination in Worked Exercise C1.

$$\begin{aligned} 2x + 4y &= 10 \\ 4x + y &= 6 \end{aligned}$$

You saw in Subsection 3.4 that such systems may be expressed in matrix form as

$$\begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \end{pmatrix}.$$

In Exercise C30(a) you found that this coefficient matrix is invertible:

$$\begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}.$$

Multiplying both sides of the matrix form of the system on the left by the inverse of the coefficient matrix, we obtain

$$\begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} 10 \\ 6 \end{pmatrix},$$

that is,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

So the system has the unique solution $x = 1$, $y = 2$.

In general, suppose that $\mathbf{Ax} = \mathbf{b}$ is the matrix form of a system of linear equations, and that the coefficient matrix $\mathbf{A}$ is invertible. Then we can multiply both sides of the equation $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{A}^{-1}$ to yield $\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$; that is, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. It seems, then, that the system has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

However, we have to be careful before making this claim. Whenever we manipulate an equation in order to solve it, we have to be sure that the manipulation yields a second equation *equivalent* to the first (otherwise the two equations might have different solution sets).

In this case, we have to be sure that

$$\mathbf{Ax} = \mathbf{b} \quad \text{if and only if} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

We showed above that if $\mathbf{Ax} = \mathbf{b}$, then multiplying both sides on the left by $\mathbf{A}^{-1}$ yields $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$; in other words, we proved that $\mathbf{Ax} = \mathbf{b}$ implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. It remains to prove that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ implies $\mathbf{Ax} = \mathbf{b}$, and fortunately this is just as easy: if $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, then multiplying both sides of this equation on the left by $\mathbf{A}$ yields $\mathbf{Ax} = \mathbf{AA}^{-1}\mathbf{b}$; that is, $\mathbf{Ax} = \mathbf{b}$.

So multiplying both sides of $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{A}^{-1}$ *does* yield an equivalent equation. We have proved the following theorem.

Theorem C8

Let $\mathbf{A}$ be an invertible matrix. Then the system of linear equations $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Exercise C31

Use Theorem C8 to solve the following system of linear equations.

$$\begin{aligned} x + y + 2z &= 1 \\ -x \phantom{{}+ y} - 4z &= 2 \\ 3x + 2y + 10z &= -1 \end{aligned}$$

In Worked Exercise C16 you saw that

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -4 \\ 3 & 2 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

In general, it is worth using the method of Theorem C8 only if we have already calculated the inverse of the coefficient matrix. To use the method of Theorem C8 to solve $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ invertible matrix, we first invert $\mathbf{A}$. This involves row-reducing the matrix $(\mathbf{A} \mid \mathbf{I})$. We then calculate the matrix product $\mathbf{A}^{-1}\mathbf{b}$. On the other hand, the method of Section 2 using Gauss–Jordan elimination involves only row-reducing the matrix $(\mathbf{A} \mid \mathbf{b})$ and so is usually quicker.

Theorem C8 shows, in particular, that if the coefficient matrix $\mathbf{A}$ of a system of linear equations $\mathbf{Ax} = \mathbf{b}$ is invertible, then the system has a *unique* solution. The converse of this result is also true – we prove this in the next theorem.

This theorem gives some important relationships between the invertibility of a matrix and the number of solutions of a system of linear equations that has this matrix as its coefficient matrix. The theorem states that three conditions are *equivalent*: any one of the conditions implies any other one.

Theorem C9

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) The system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each $n \times 1$ matrix $\mathbf{b}$.

(c) The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

Proof We show that (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (a), which shows that the conditions are equivalent.

(a) $\Longrightarrow$ (b)
Suppose that $\mathbf{A}$ is an invertible $n \times n$ matrix. Then, by Theorem C8, for any $n \times 1$ matrix $\mathbf{b}$, the system $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

(b) $\Longrightarrow$ (c)
Suppose that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each $n \times 1$ matrix $\mathbf{b}$. Then, in particular, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has a unique solution. But every homogeneous system has the trivial solution; thus this unique solution must be the trivial one.

(c) $\Longrightarrow$ (a)
Suppose that the system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution. Then row-reducing the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right)$$

of the system must yield

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right),$$

since this is the row-reduced matrix that corresponds to each unknown being 0. If we now ignore the last column of each of the matrices appearing in this row-reduction, we are left with a reduction of $\mathbf{A}$ to $\mathbf{I}$. Hence, by the Invertibility Theorem (Theorem C7), $\mathbf{A}$ is invertible. ■

4.4 Elementary matrices

In this subsection you will meet a class of square matrices associated with elementary row operations and investigate their properties.

We will use these matrices and their properties in Subsection 4.5 to help prove the Invertibility Theorem (Theorem C7). We will also find them useful later.

Consider the following matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

They are obtained by performing, on the 3×3 identity matrix, the elementary row operations $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$, $\mathbf{r}_2 \to 5\mathbf{r}_2$ and $\mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2$, respectively.

Definition

A matrix obtained by performing an elementary row operation on an identity matrix is an **elementary matrix**.

The elementary row operation that is performed to obtain an elementary matrix from an identity matrix is called the elementary row operation *associated with* that elementary matrix.

We now demonstrate the most important property of elementary matrices. Below, we show the effect of multiplying the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

on the left by each of the above elementary matrices. Notice that in each case, the resulting matrix is precisely the matrix that is obtained when the row operation associated with the elementary matrix is performed on $\mathbf{A}$.

$$\underset{\substack{\text{elementary matrix}\\ \text{associated with}\\ \mathbf{r}_1 \leftrightarrow \mathbf{r}_2}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}} = \underset{\substack{\text{matrix obtained when}\\ \mathbf{r}_1 \leftrightarrow \mathbf{r}_2\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{pmatrix}}$$

$$\underset{\substack{\text{elementary matrix}\\ \text{associated with}\\ \mathbf{r}_2 \to 5\mathbf{r}_2}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}} = \underset{\substack{\text{matrix obtained when}\\ \mathbf{r}_2 \to 5\mathbf{r}_2\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 25 & 30 & 35 & 40 \\ 9 & 10 & 11 & 12 \end{pmatrix}}$$

$$\underset{\substack{\text{elementary matrix}\\ \text{associated with}\\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}} \quad \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}} = \underset{\substack{\text{matrix obtained when}\\ \mathbf{r}_3 \to \mathbf{r}_3 + 2\mathbf{r}_2\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 19 & 22 & 25 & 28 \end{pmatrix}}$$

There is nothing special about the above elementary matrices, or about the above matrix $\mathbf{A}$. In the next exercise you will find that other elementary matrices behave similarly.

Exercise C32

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$.

(a) Write down the 2×2 elementary matrix associated with the elementary row operation $\mathbf{r}_1 \to 5\mathbf{r}_1$.

Multiply $\mathbf{A}$ on the left by this elementary matrix, and check that the resulting matrix is the same as the matrix obtained when $\mathbf{r}_1 \to 5\mathbf{r}_1$ is performed on $\mathbf{A}$.

(b) Write down the 4×4 elementary matrix associated with the elementary row operation $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4$.

Multiply $\mathbf{B}$ on the left by this elementary matrix, and check that the resulting matrix is the same as the matrix obtained when $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4$ is performed on $\mathbf{B}$.

Notice that the number of columns of the elementary matrix used must equal the number of rows of the matrix upon which the elementary operation is to be performed; that is, the elementary row operations should be applied to an appropriately sized identity matrix to obtain the elementary matrix required.

In general, we have the following theorem, which we state without proof.

Theorem C10

Let $\mathbf{E}$ be an elementary matrix, and let $\mathbf{A}$ be any matrix with the same number of rows as $\mathbf{E}$. Then the product $\mathbf{EA}$ is the same as the matrix obtained when the elementary row operation associated with $\mathbf{E}$ is performed on $\mathbf{A}$.

Theorem C10 tells us that if we perform an elementary row operation on a matrix $\mathbf{A}$ with m rows, then the resulting matrix is $\mathbf{EA}$, where $\mathbf{E}$ is the $m \times m$ elementary matrix associated with the row operation.

What happens if we perform a *sequence* of k elementary row operations on a matrix $\mathbf{A}$ with m rows? Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $m \times m$ elementary matrices associated with the row operations in the sequence, in the same order. The first row operation is performed on $\mathbf{A}$, producing the matrix $\mathbf{E}_1\mathbf{A}$; the second row operation is then performed on *this* matrix, producing the matrix $\mathbf{E}_2(\mathbf{E}_1\mathbf{A}) = \mathbf{E}_2\mathbf{E}_1\mathbf{A}$; and so on. After the whole sequence of k row operations has been performed, the resulting matrix is $\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$. Notice that the order of the elementary matrices in this matrix product is the *reverse* of the order in which their associated row operations are performed.

This fact will be useful later, and we record it as a corollary to Theorem C10.

Corollary C11

Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $m \times m$ elementary matrices associated with a sequence of k elementary row operations carried out on a matrix $\mathbf{A}$ with m rows, in the same order. Then, after the sequence of row operations has been performed, the resulting matrix is

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}.$$

For example, earlier, to illustrate the Invertibility Theorem (Theorem C7), we performed the sequence of row operations

$$\mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1, \quad \mathbf{r}_2 \to \tfrac{1}{3}\mathbf{r}_2, \quad \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2,$$

on the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}$$

to produce the identity matrix $\mathbf{I}_2$.

By Corollary C11 we have the following, which you should check by evaluating the product on the right-hand side.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 9 \end{pmatrix}.$$

We now explore some other useful connections between elementary row operations and elementary matrices. We begin by introducing a further property of elementary row operations.

In the following example, the second elementary row operation undoes the effect of the first.

$$\begin{array}{ll} \mathbf{r}_1 & \multirow{2}{*}{$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$} \\ \mathbf{r}_2 & \end{array}$$

$$\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1 \quad \begin{pmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \end{pmatrix}$$

$$\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1 \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

In fact, given any elementary row operation, it is easy to write down an *inverse* elementary row operation that undoes the effect of the first, as summarised in the following table.

Elementary row operation	Inverse elementary row operation
$\mathbf{r}_i \leftrightarrow \mathbf{r}_j$	$\mathbf{r}_i \leftrightarrow \mathbf{r}_j$
$\mathbf{r}_i \to c\,\mathbf{r}_i \quad (c \neq 0)$	$\mathbf{r}_i \to (1/c)\,\mathbf{r}_i$
$\mathbf{r}_i \to \mathbf{r}_i + c\,\mathbf{r}_j$	$\mathbf{r}_i \to \mathbf{r}_i - c\,\mathbf{r}_j$

Exercise C33

Write down the inverse of each of the following elementary row operations. Check your answer in each case by carrying out the sequence of two row operations on the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

(a) $\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2$ (b) $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ (c) $\mathbf{r}_2 \to -3\mathbf{r}_2$

Note that if two elementary row operations are such that the second is the inverse of the first, then the first is the inverse of the second – so it makes sense to say that they are *inverses of each other*, or that they form an **inverse pair**. For example, the inverse of $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$ is $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$, and the inverse of $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$ is $\mathbf{r}_2 \to \mathbf{r}_2 - (-3)\mathbf{r}_1$, that is, $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$. So $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$ and $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$ are inverses of each other.

Now consider the following pair of 2×2 elementary matrices associated with the inverse pair of elementary row operations $\mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_1$ and $\mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_1$:

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

These two matrices are themselves inverses of each other, as we can easily check:

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

This connection between inverse pairs of elementary row operations and inverse pairs of elementary matrices holds in general.

Theorem C12

Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be elementary matrices of the same size whose associated elementary row operations are inverses of each other. Then $\mathbf{E}_1$ and $\mathbf{E}_2$ are inverses of each other.

Proof In this proof we refer to the row operations associated with $\mathbf{E}_1$ and $\mathbf{E}_2$ as row operation 1 and row operation 2, respectively.

By Corollary C11, $\mathbf{E}_2\mathbf{E}_1\mathbf{I}$ is the matrix produced when row operations 1 and 2 are performed, in that order, on $\mathbf{I}$. Similarly, $\mathbf{E}_1\mathbf{E}_2\mathbf{I}$ is the matrix produced when row operations 2 and 1 are performed, in that order, on $\mathbf{I}$. But each of these two row operations undoes the effect of the other, so $\mathbf{E}_2\mathbf{E}_1\mathbf{I} = \mathbf{I}$ and $\mathbf{E}_1\mathbf{E}_2\mathbf{I} = \mathbf{I}$; that is,

$$\mathbf{E}_2\mathbf{E}_1 = \mathbf{I} = \mathbf{E}_1\mathbf{E}_2.$$

Thus $\mathbf{E}_1$ and $\mathbf{E}_2$ are inverses of each other. ■

Theorem C12 has the following corollary.

Corollary C13

Every elementary matrix is invertible, and its inverse is also an elementary matrix.

Proof Let $\mathbf{E}$ be an elementary matrix. Then $\mathbf{E}$ has an associated elementary row operation. This associated elementary row operation has an inverse operation, and the elementary matrix of the same size as $\mathbf{E}$ associated with this inverse operation is the inverse of $\mathbf{E}$, by Theorem C12. ■

Exercise C34

Use the method suggested by the proof of Corollary C13 to find the inverse of the elementary matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.5 Proof of the Invertibility Theorem

We are now ready to prove the Invertibility Theorem, using elementary matrices and their properties. We first remind you of the theorem.

Theorem C7 Invertibility Theorem

(a) A square matrix is invertible if and only if its row-reduced form is $\mathbf{I}$.

(b) Any sequence of elementary row operations that transforms a matrix $\mathbf{A}$ to $\mathbf{I}$ also transforms $\mathbf{I}$ to $\mathbf{A}^{-1}$.

Proof Let $\mathbf{A}$ be an $n \times n$ matrix, and let the row-reduced form of $\mathbf{A}$ be $\mathbf{U}$. Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ be the $n \times n$ elementary matrices associated with a sequence of k elementary row operations that transforms $\mathbf{A}$ to $\mathbf{U}$. Then, by Corollary C11,

$$\mathbf{U} = \mathbf{BA},$$

where $\mathbf{B} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$. Now $\mathbf{B}$ is invertible – since every elementary matrix is invertible (by Corollary C13), and a product of invertible matrices is invertible (by Theorem C5).

(a) We start by proving the *only if* statement.

First we show that if $\mathbf{A}$ is invertible, then $\mathbf{U} = \mathbf{I}$.

Suppose that $\mathbf{A}$ is invertible. Then $\mathbf{U}$ is a product of invertible matrices ($\mathbf{B}$ and $\mathbf{A}$); hence $\mathbf{U}$ is invertible.

Therefore $\mathbf{U}$ does not have a zero row (since, by Theorem C4, a square matrix with a zero row is not invertible), and so from the definition of row-reduced form, it has a leading 1 in each of its n rows. Each of these n leading ones lies in a different column; so, since $\mathbf{U}$ has only n columns, each column must contain a leading 1. Thus the leading 1 in the top row must lie in the left-most position, and the leading 1 in each subsequent row must lie just one position to the right of the leading 1 in the row immediately above. All the entries above and below these leading ones are zeros. Hence $\mathbf{U} = \mathbf{I}$.

We now prove the *if* statement.

Next, we show that if $\mathbf{U} = \mathbf{I}$, then $\mathbf{A}$ is invertible.

Suppose that $\mathbf{U} = \mathbf{I}$. Then

$$\mathbf{I} = \mathbf{BA}. \tag{5}$$

Multiplying both sides of equation (5) on the left by $\mathbf{B}^{-1}$ yields

$$\mathbf{B}^{-1}\mathbf{I} = \mathbf{B}^{-1}\mathbf{B}\mathbf{A},$$

that is,

$$\mathbf{B}^{-1} = \mathbf{A}.$$

Multiplying both sides of *this* equation on the right by $\mathbf{B}$ yields

$$\mathbf{B}^{-1}\mathbf{B} = \mathbf{A}\mathbf{B},$$

that is,

$$\mathbf{I} = \mathbf{A}\mathbf{B}. \tag{6}$$

Equations (5) and (6) together tell us that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \mathbf{B}$.

(b) It follows from the proof of part (a) that if $\mathbf{U} = \mathbf{I}$, then $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \mathbf{B}$; that is, $\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$.

This equation can be written as

$$\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{I},$$

which tells us that $\mathbf{A}^{-1}$ is the matrix produced by performing on $\mathbf{I}$ the sequence of row operations associated with $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$. ■

5 Determinants

In this section you will revise the *determinant* of a 2×2 matrix, and see how this concept extends to $n \times n$ matrices.

5.1 Systems of linear equations and determinants

Determinants arise naturally in the study of systems of linear equations.

In 1693 Gottfried Wilhelm Leibniz (1646–1716) wrote a letter to the Marquis de l'Hôpital in which he demonstrated a method for solving a system of three simultaneous equations which involved calculating what we now call the determinant of a 3×3 matrix, and went on to give a general (although rather unclear) rule for calculating the determinant of an $n \times n$ matrix.

The actual term 'determinant' was introduced by Carl Friedrich Gauss (1777–1855) in his *Disquistiones Arithmeticae* of 1801, but it was Augustin-Louis Cauchy (1789–1857) who in 1812, adapting the term from Gauss, first used it in its modern sense and began to develop a proper theory of determinants.

Gabriel Cramer

Colin Maclaurin

This connection between determinants and systems of linear equations was made explicit by Gabriel Cramer in a method known as *Cramer's rule*. If a unique solution exists for a system of n linear equations in n unknowns, then this solution can be found by evaluating determinants. You will see Cramer's rule for a system of two linear equations in two unknowns; it is rather unwieldy to use for larger systems. However, Cramer's rule gives an expression for each unknown individually, so it makes it possible to find one unknown without solving the whole system.

Cramer's rule is named after the Swiss mathematician Gabriel Cramer (1704–1752) who presented it in his *Introduction à l'analyse des lignes courbes algébriques* (*Introduction to the Analysis of Algebraic Curved Lines*) of 1750, although the Scottish mathematician Colin Maclaurin (1698–1746) had already described the rule in his *Treatise of Algebra* (1748) written in 1730 but not published until after his death.

Determinant of a 2×2 matrix

We start by looking at a system of two equations in two unknowns, where the coefficients of the system are real numbers.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{or} \quad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Using Gauss–Jordan elimination, the following solution can be found,

$$x = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \quad y = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}, \tag{7}$$

provided that $a_1b_2 - b_1a_2$ is not zero. (You can check this solution by substitution.) We call the expression $a_1b_2 - b_1a_2$ the *determinant* of the coefficient matrix. Each term in this expression contains the letters a and b, and the subscripts 1 and 2, in some order.

The definition we give for the determinant of a 2×2 matrix is in a form that is easier to remember.

Definition

The **determinant** of a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

You might find it helpful to remember the scheme shown in Figure 15.

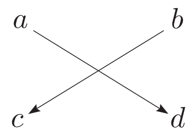

Figure 15 Scheme for 2×2 determinant

We write $\det \mathbf{A}$, and use vertical bars '$| \ldots |$' around the matrix entries, in place of the round brackets, to denote the determinant. Some texts use the notation $|\mathbf{A}|$ rather than $\det \mathbf{A}$.

For example, let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; then

$$\det \mathbf{A} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 3) = -2.$$

Exercise C35

Evaluate the determinant of each of the following matrices.

(a) $\begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 3 \\ 17 & 2 \end{pmatrix}$

Notice that the numerators of the solutions for x and y in (7) can also be written as determinants:

$$c_1b_2 - b_1c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad a_1c_2 - c_1a_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

So we could write these solutions as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

This is Cramer's rule for a system of two linear equations in two unknowns. The numerator of the expression for x is the determinant of the coefficient matrix of the system with the first column replaced by the constant terms. Similarly, the numerator of the expression for y is the determinant of the coefficient matrix of the system with the second column replaced by the constant terms.

In Subsection 5.4 we will prove that a 2×2 matrix is invertible if and only if its determinant is non-zero. For an invertible 2×2 matrix, there is a quick way to find the inverse using the determinant. You can verify the following strategy by checking that the expression given below for $\mathbf{A}^{-1}$ does indeed satisfy $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

Strategy C4

To find the inverse of a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det \mathbf{A} = ad - bc \neq 0$:

- interchange the diagonal entries
- multiply the non-diagonal entries by -1
- divide by the determinant of $\mathbf{A}$, giving

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercise C36

Determine whether or not each of the following matrices is invertible, and find the inverse if it exists.

(a) $\begin{pmatrix} 4 & 2 \\ 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

There is also a geometric interpretation of the determinant: let (a, c) and (b, d) be two position vectors. Then the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

gives the area of the parallelogram with adjacent sides given by these position vectors. For example, the parallelogram shown in Figure 16 with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$ and $(3, 4)$ has area 5, since the base and height are both equal to $\sqrt{5}$. Now, since one of the vertices is at the origin, the position vectors $(2, 1)$ and $(1, 3)$ determine the parallelogram, and

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = (2 \times 3) - (1 \times 1) = 5.$$

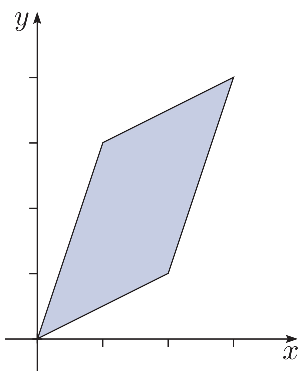

Figure 16 A parallelogram with area 5

Determinant of a 3×3 matrix

We now consider the following system of three linear equations in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \quad \text{or} \quad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Again we can find the solution, if one exists, using Gauss–Jordan elimination. It turns out that the solutions for x, y and z all have the same denominator:

$$a_1b_2c_3 - a_1c_2b_3 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3.$$

This is the *determinant* of the 3×3 coefficient matrix. Notice that each term in this expression for the denominator contains the letters a, b and c, and the subscripts 1, 2 and 3, in some order.

The definition we give for the determinant of a 3×3 matrix is expressed in terms of three 2×2 determinants. This is the easiest way to remember the definition.

Definition

The **determinant** of a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

is

$$\det \mathbf{A} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notice the minus sign before the second term on the right-hand side.

Worked Exercise C17

Evaluate the determinant of each of the following 3×3 matrices.

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

Solution

(a)
$$\begin{aligned} \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} \\ &= 1\,((1 \times 1) - (-1 \times 1)) \\ &\quad - 2\,((3 \times 1) - (-1 \times (-2))) \\ &\quad + 1\,((3 \times 1) - (1 \times (-2))) \\ &= 5 \end{aligned}$$

(b)
$$\begin{aligned} \begin{vmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix} &= 4 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} \\ &= 4\,((-1 \times 3) - (2 \times 1)) \\ &\quad - 0 + 1\,((0 \times 1) - (-1 \times 2)) \\ &= -18 \end{aligned}$$

Exercise C37

Evaluate the determinant of each of the following 3×3 matrices.

(a) $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 10 & 0 \\ 3 & -1 & 2 \\ 5 & 9 & 2 \end{pmatrix}$

Determinants of larger matrices (4×4, and so on) are defined similarly in terms of smaller determinants in the next subsection. Note that determinants are defined only for square matrices. As with 2×2 matrices, determinants of larger matrices can be used to solve systems of linear equations.

5.2 Evaluating determinants

You have seen that although the determinant of a 2×2 matrix is simple to evaluate, the determinant of a 3×3 matrix is quite complicated. Determinants of larger matrices become increasingly more complicated as the size of the matrix increases. You will mainly be finding determinants of matrices of size 2×2 and 3×3. In this subsection we develop a strategy for evaluating determinants by expressing them eventually in terms of determinants of 2×2 matrices, as with the definition of the determinant of a 3×3 matrix above.

Cofactors

A **submatrix** is a matrix formed from another matrix with some of the rows and/or columns removed; submatrices are useful when evaluating determinants.

We can express the determinant of a 3×3 matrix $\mathbf{A} = (a_{ij})$ conveniently as

$$\det \mathbf{A} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

The elements A_{11}, A_{12} and A_{13} in this expression are called the *cofactors* of the elements a_{11}, a_{12} and a_{13}, respectively. We can see from the definition of the determinant that these cofactors are themselves determinants with a $+$ or $-$ sign attached. In fact, A_{1j} is $(-1)^{1+j}$ times the determinant of a submatrix of $\mathbf{A}$ formed by removing the top row and one column of $\mathbf{A}$ – namely the row and column containing the element a_{1j}.

Thus for A_{11} we have

$$(-1)^{1+1}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{so} \quad A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

for A_{12} we have

$$(-1)^{1+2}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{so} \quad A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

and for A_{13} we have

$$(-1)^{1+3}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{so} \quad A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

In fact, there is a *cofactor* associated with each entry of any square matrix.

Definition

Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix. The **cofactor** A_{ij} associated with the entry a_{ij} is

$$A_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij},$$

where $\mathbf{A}_{ij}$ is the $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ resulting when the ith row and jth column (the row and column containing the entry a_{ij}) are removed.

For example, for the cofactor A_{23} of the 4×4 matrix $\mathbf{A} = (a_{ij})$ we have

$$(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \quad \text{so} \quad A_{23} = -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}.$$

Exercise C38

Write down expressions for the cofactors A_{13} and A_{45} of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

(Do not attempt to evaluate these expressions!)

Determinant of an $n \times n$ matrix

You have seen that we can use cofactors to evaluate the determinant of a 3×3 matrix. Determinants of larger matrices can be evaluated in a similar way.

Definition

The **determinant** of an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ is

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$
$$= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$$

Do not forget the minus sign that is a part of alternate cofactors!

The determinant of a matrix is a complicated string of terms. The definition above collects the terms into manageable expressions using the cofactors of the entries of the top row; when we write the determinant in this way, we say that we are *expanding along the top row*.

There are alternative expansions for the determinant of a square matrix that collect the terms in different ways – however, the resulting value for the determinant is always the same.

We are now in a position to evaluate the determinant of any square matrix using the following strategy.

Strategy C5

To evaluate the determinant of an $n \times n$ matrix:

1. expand along the top row to express the $n \times n$ determinant in terms of n determinants of size $(n-1) \times (n-1)$
2. expand along the top row of each of the resulting determinants
3. repeatedly apply step 2 until the only determinants in the expression are of size 2×2
4. evaluate the final expression.

Figure 17 gives a scheme for an $n \times n$ determinant.

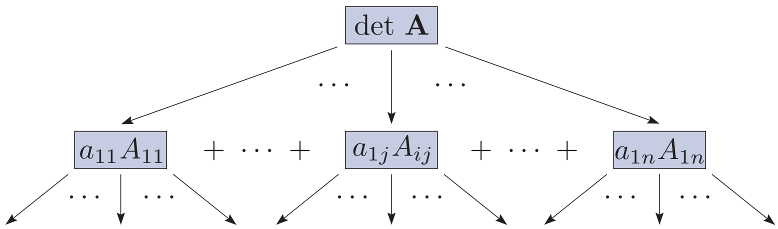

Figure 17 Scheme for an $n \times n$ determinant

Worked Exercise C18 illustrates Strategy C5, before you are asked to find the determinant of a 4×4 matrix in Exercise C39.

Worked Exercise C18

Evaluate the following determinant.

$$\begin{vmatrix} 2 & 0 & 3 & 5 \\ 0 & 4 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}$$

Solution

We apply Strategy C5:

$$\begin{aligned}
\begin{vmatrix} 2 & 0 & 3 & 5 \\ 0 & 4 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix} &= 2\begin{vmatrix} 4 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} - 0 + 3\begin{vmatrix} 0 & 4 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} - 5\begin{vmatrix} 0 & 4 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix} \\
&= 2\left(4\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 0\right) \\
&\quad + 3\left(0 - 4\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0\right) \\
&\quad - 5\left(0 - 4\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\right) \\
&= 2(-4-2) + 3(-4) - 5(-4-2) \\
&= -12 - 12 + 30 \\
&= 6.
\end{aligned}$$

Exercise C39

Evaluate the following determinant.

$$\begin{vmatrix} 0 & 2 & 1 & -1 \\ -3 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{vmatrix}$$

5.3 Properties of determinants

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ matrices. Are there any relationships between $\det \mathbf{A}$, $\det \mathbf{B}$, $\det(\mathbf{A}+\mathbf{B})$ and $\det(\mathbf{AB})$?

Exercise C40

Let $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix}$.

Evaluate $\det \mathbf{A}$, $\det \mathbf{B}$, $\det(\mathbf{A}+\mathbf{B})$, $\det(\mathbf{AB})$ and $(\det \mathbf{A})(\det \mathbf{B})$.

Comment on your results.

You should have found in the solution to Exercise C40 that there does not appear to be a simple relationship for the addition of determinants; that is, we cannot easily express $\det(\mathbf{A}+\mathbf{B})$ in terms of $\det \mathbf{A}$ and $\det \mathbf{B}$. However, the identity

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$$

does hold for all square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size. The simplicity of this result is somewhat surprising, given the complexity of the definitions of matrix multiplication and the determinant.

We collect together, without proof, some results about determinants in the following theorem.

Theorem C14

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same size. Then the following hold:

(a) $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$

(b) $\det \mathbf{I} = 1$

(c) $\det \mathbf{A}^T = \det \mathbf{A}$.

Elementary operations and determinants

Earlier, in Theorem C10, you saw that multiplication on the left by an *elementary matrix* has the same effect as applying the associated elementary row operation. Here, we use elementary matrices to prove some useful results about determinants.

Exercise C41

Evaluate the following determinants, where k is any real number.

(a) $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix}$

The results of Exercise C41 are particular cases of the following theorem. The proof is not hard, but it is not very illuminating, so is not given here.

Theorem C15

Let $\mathbf{E}$ be an elementary matrix, and let k be a non-zero real number.

(a) If $\mathbf{E}$ results from interchanging two rows of $\mathbf{I}$, then $\det \mathbf{E} = -1$.

(b) If $\mathbf{E}$ results from multiplying a row of $\mathbf{I}$ by k, then $\det \mathbf{E} = k$.

(c) If $\mathbf{E}$ results from adding k times one row of $\mathbf{I}$ to another row, then $\det \mathbf{E} = 1$.

Zeros in a matrix greatly simplify the calculation of the determinant. If an entire row of the matrix is zero, then all the terms vanish and the determinant is zero. Some other matrices with zero determinant are also easy to recognise.

Theorem C16

Let $\mathbf{A}$ be a square matrix. Then $\det \mathbf{A} = 0$ if any of the following hold:

(a) $\mathbf{A}$ has an entire row (or column) of zeros

(b) $\mathbf{A}$ has two equal rows (or columns)

(c) $\mathbf{A}$ has two proportional rows (or columns).

Proof We prove the statements for rows. The results for columns follow, as Theorem C14(c) states that taking the transpose does not alter the determinant of a matrix.

(a) We follow Strategy C5 and expand along the top row of $\mathbf{A}$, and continue by expanding along the top row of the resulting determinants until the only determinants in the expression are of size 2×2. The first term of the full expansion is therefore the product $a_{11}a_{22}a_{33} \cdots a_{nn}$, and each other term similarly comprises a product containing one entry from each row and each column.

Each term in the full expansion of the determinant of $\mathbf{A}$ is a product containing one entry from each row and each column of $\mathbf{A}$. If an entire row of $\mathbf{A}$ is zero, then each term of this expansion contains at least one zero, and so each term is zero. Hence the determinant of $\mathbf{A}$ is equal to zero.

(b) If the ith and jth rows of the matrix $\mathbf{A}$ are equal, then $\mathbf{A}$ remains the same if these rows are interchanged. Let $\mathbf{E}$ be the elementary matrix obtained by interchanging the ith and jth rows of $\mathbf{I}$. Then $\mathbf{EA} = \mathbf{A}$. Using Theorems C14 and C15, we have

$$\det \mathbf{A} = \det(\mathbf{EA}) = (\det \mathbf{E})(\det \mathbf{A}) = -1 \times \det \mathbf{A}.$$

This implies that $\det \mathbf{A} = 0$, as required.

(c) Two rows (or columns) of a matrix are *proportional* when one is a multiple of the other.

Suppose that the ith row of $\mathbf{A}$ is equal to k times the jth row. Let $\mathbf{E}$ be the elementary matrix obtained from $\mathbf{I}$ by multiplying the ith row by $1/k$. Then the ith and jth rows of the matrix $\mathbf{EA}$ are equal. The determinant of this matrix $\mathbf{EA}$ is zero, by (b) above. Using Theorem C14, we have

$$(\det \mathbf{E})(\det \mathbf{A}) = \det(\mathbf{EA}) = 0.$$

Now $\det \mathbf{E} = 1/k$, by Theorem C15. This implies that $\det \mathbf{A} = 0$, as required. ■

Exercise C42

Evaluate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 13 & 11 \\ -2 & 4 & -8 \end{pmatrix}.$$

Theorem C15(a) and Theorem C14(a) together mean that if $\mathbf{B}$ is a matrix obtained from a matrix $\mathbf{A}$ by interchanging a pair of rows, then $\det \mathbf{B} = -\det \mathbf{A}$. Therefore the evaluation of the determinant can be significantly simplified if a row of the matrix contains some zeros, as the following worked exercise illustrates.

Worked Exercise C19

Evaluate the determinant of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 7 & 0 \\ 3 & 1 & 5 \end{pmatrix}$$

Solution

The second row of $\mathbf{A}$ has only one non-zero entry and so interchanging the top two rows will give only one non-zero term in the expansion.

We interchange the top two rows of $\mathbf{A}$, and apply Theorems C14 and C15, giving

$$\det \mathbf{A} = \begin{vmatrix} 1 & 4 & 2 \\ 0 & 7 & 0 \\ 3 & 1 & 5 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 7 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix}.$$

We use Strategy C5 to evaluate this determinant:

$$\begin{aligned} \det \mathbf{A} &= (-1) \left(0 - 7 \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} + 0 \right) \\ &= 7\left((1 \times 5) - (2 \times 3)\right) \\ &= -7. \end{aligned}$$

Exercise C43

Evaluate the determinant of the following matrix.

$$\mathbf{A} = \begin{pmatrix} 10 & 3 & -4 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

5.4 Determinants and inverses of matrices

Earlier, in Subsection 5.1, we stated that the inverse of a 2×2 matrix $\mathbf{A}$ exists if and only if $\det \mathbf{A} \neq 0$. This extends to all square matrices.

Theorem C17

A square matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.

Proof Let $\mathbf{A}$ be an $n \times n$ matrix.

We start by proving the *only if* statement.

First we show that if $\mathbf{A}$ is invertible, then $\det \mathbf{A} \neq 0$.

Suppose that $\mathbf{A}$ is invertible. Then since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$, it follows from Theorem C14 that

$$(\det \mathbf{A})(\det \mathbf{A}^{-1}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det \mathbf{I}_n = 1.$$

Therefore neither $\det \mathbf{A}$ nor $\det \mathbf{A}^{-1}$ is 0.

We now prove the *if* statement.

Next we show that if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is invertible.

Now suppose that $\det \mathbf{A} \neq 0$. Let $\mathbf{E}_1, \ldots, \mathbf{E}_k$ be elementary matrices such that $\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}$ is matrix $\mathbf{A}$ in row-reduced form. Using Theorems C14 and C15 and the assumption that $\det \mathbf{A} \neq 0$, we have

$$\det \mathbf{U} = (\det \mathbf{E}_k) \cdots (\det \mathbf{E}_2)(\det \mathbf{E}_1)(\det \mathbf{A}) \neq 0.$$

Now this implies that $\mathbf{U}$ has no zero row, and therefore has a leading 1 in each of its n rows. Hence $\mathbf{U} = \mathbf{I}_n$, and so, by the Invertibility Theorem (Theorem C7), the matrix $\mathbf{A}$ is invertible, with

$$\mathbf{A}^{-1} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1.$$

■

We saw in the proof of Theorem C17 above that if $\mathbf{A}$ is invertible, then $(\det \mathbf{A})(\det \mathbf{A}^{-1}) = 1$. This gives the following useful result.

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}$$

Until now, if we wanted to show that an $n \times n$ matrix $\mathbf{A}$ is invertible, we had to produce an $n \times n$ matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}.$$

The next theorem shows that if one of these conditions holds, then the other holds automatically. Thus if we want to show that an $n \times n$ matrix $\mathbf{A}$ is invertible, it is enough to produce an $n \times n$ matrix $\mathbf{B}$ satisfying *either* condition.

Theorem C18

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. Then $\mathbf{AB} = \mathbf{I}$ if and only if $\mathbf{BA} = \mathbf{I}$.

Proof We start by proving the *only if* statement.

First we show that if $\mathbf{AB} = \mathbf{I}$, then $\mathbf{BA} = \mathbf{I}$.

Suppose that $\mathbf{AB} = \mathbf{I}$. Then, by Theorem C14,

$$(\det \mathbf{A})(\det \mathbf{B}) = \det(\mathbf{AB}) = \det \mathbf{I} = 1.$$

This implies that

$$\det \mathbf{A} \neq 0 \quad \text{and} \quad \det \mathbf{B} \neq 0,$$

so, by Theorem C17, $\mathbf{A}$ and $\mathbf{B}$ are both invertible.

Now,

$$\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{I},$$

and we can write $\mathbf{I}$ as $\mathbf{AB}$, so

$$\mathbf{A}^{-1} = \mathbf{A}^{-1}(\mathbf{AB}) = (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{IB} = \mathbf{B},$$

and therefore

$$\mathbf{BA} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

as required.

To prove the *if* statement we have to show that if $\mathbf{BA} = \mathbf{I}$, then $\mathbf{AB} = \mathbf{I}$. We can use exactly the same argument as above with $\mathbf{A}$ and $\mathbf{B}$ exchanged.

The same argument, with the roles of $\mathbf{A}$ and $\mathbf{B}$ interchanged, proves the converse. ■

We summarise the results on the invertibility of a matrix $\mathbf{A}$ as follows. This one theorem collects together Theorems C7, C9 and C17.

Theorem C19 Summary Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following statements are equivalent.

(a) $\mathbf{A}$ is invertible.

(b) $\det \mathbf{A} \neq 0$.

(c) The row-reduced form of $\mathbf{A}$ is $\mathbf{I}_n$.

(d) The system $\mathbf{Ax} = \mathbf{b}$ has precisely one solution for each $n \times 1$ matrix $\mathbf{b}$.

(e) The system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

To conclude this section, we collect together some of the most important properties of matrices from this unit.

Summary of properties of matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices of the same size. Then

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}),$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T,$$

$$\det \mathbf{A}^T = \det \mathbf{A}.$$

If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1},$$

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}.$$

Summary

In this unit you have seen that systems of linear equations can have no solution, a unique solution or infinitely many solutions, and you have used Gauss–Jordan elimination to solve such systems. You have seen that matrices can be used in two different ways to represent systems of linear equations: both as an augmented matrix and as a matrix equation in which the coefficient matrix is multiplied on the right by the matrix of unknowns to give the matrix of constant terms. You studied how properties of matrices relate to properties of the corresponding systems of linear equations. In particular, you saw that if the coefficient matrix of a system of linear equations is invertible, or equivalently, if the determinant of the coefficient matrix is non-zero, then the system has a unique solution. You also saw that the set of $m \times n$ matrices with real entries forms an abelian group under addition and that the set of $n \times n$ invertible matrices with real entries forms an abelian group under matrix multiplication.

You will encounter systems of linear equations throughout the linear algebra units, along with matrices and their properties. Matrices will also appear in the group theory units, in particular you will work with the group of invertible 2×2 matrices in Book E.

Learning outcomes

After working through this unit, you should be able to:

- understand the connection between the solutions of systems of linear equations in two and three unknowns, and the intersection of lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$
- explain the terms *solution set*, *consistent*, *inconsistent* and *homogeneous system of linear equations*
- use the method of *Gauss–Jordan elimination* to find the solutions of systems of linear equations
- describe the three types of *elementary operation* and *elementary row operation*
- recognise whether or not a given matrix is in *row-reduced form* and row-reduce a matrix
- write down the *augmented matrix* of a system of linear equations, recover a system of linear equations from its augmented matrix, and solve a system of linear equations by row-reducing its augmented matrix
- perform the matrix operations of addition, multiplication and transposition
- recognise the following types of matrix: *square*, *zero*, *diagonal*, *lower triangular*, *upper triangular*, *identity*, *symmetric*
- express a system of linear equations in *matrix form* and state the relationship between the invertibility of the coefficient matrix and the number of solutions of the system
- understand what is meant by an *invertible* matrix and determine whether or not a given matrix is invertible and, if it is, find its inverse
- understand that the set of $n \times n$ invertible matrices with real entries forms a group under matrix multiplication
- understand the connections between elementary row operations and elementary matrices
- understand the term *determinant* of a square matrix, evaluate the determinant of a 2×2 matrix and *expand along the top row* to calculate the determinant of larger matrices
- use determinants to check whether or not a matrix is invertible.

Solutions to exercises

Solution to Exercise C1

(a) This is a linear equation.

(b) This is not a linear equation. The third term involves the product of x_3 and x_4.

(c) This is a linear equation (although not all of the five unknowns appear in this equation).

(d) This is not a linear equation. For example, the second term, $a_2x_2^2$, involves a product of unknowns.

Solution to Exercise C2

A general homogeneous system of m linear equations in n unknowns is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots\ \ & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

We substitute the values $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$ into the equations of the system:

$$\begin{aligned} a_{11}0 + \cdots + a_{1n}0 &= 0 \\ a_{21}0 + \cdots + a_{2n}0 &= 0 \\ \vdots \qquad\qquad \vdots\ \ & \quad \vdots \\ a_{m1}0 + \cdots + a_{mn}0 &= 0. \end{aligned}$$

All the equations are satisfied, whatever the values of the coefficients a_{ij}. The solution set therefore contains the trivial solution

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0.$$

Solution to Exercise C3

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y = 4 \\ \mathbf{r}_2 & 2x - y = 5 \end{array}$$

First we eliminate the unknown x from the second equation.

$$\begin{array}{ll} & x + y = 4 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & -3y = -3 \end{array}$$

We then simplify this equation.

$$\begin{array}{ll} & x + y = 4 \\ \mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2 & y = 1 \end{array}$$

We use it to eliminate the unknown y from the first equation of the system.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x = 3 \\ & y = 1 \end{array}$$

We conclude that there is a unique solution: $x = 3$, $y = 1$.

The method above eliminates the unknowns in order; you may have begun by performing the elementary operation $\mathbf{r}_1 \to \mathbf{r}_1 + \mathbf{r}_2$ to eliminate y from $\mathbf{r}_1$. This is also correct.

Solution to Exercise C4

The explanations in between the systems of three linear equations are not a necessary part of the solution: they are included for clarity.

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{ll} \mathbf{r}_1 & x + y - z = 8 \\ \mathbf{r}_2 & 2x - y + z = 1 \\ \mathbf{r}_3 & -x + 3y + 2z = -8 \end{array}$$

First use the $\mathbf{r}_1$ equation to eliminate the unknown x from the other equations.

$$\begin{array}{ll} & x + y - z = 8 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 & -3y + 3z = -15 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_1 & 4y + z = 0 \end{array}$$

Now simplify $\mathbf{r}_2$.

$$\begin{array}{ll} & x + y - z = 8 \\ \mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2 & y - z = 5 \\ & 4y + z = 0 \end{array}$$

Then use $\mathbf{r}_2$ to eliminate the y-terms from $\mathbf{r}_1$ and $\mathbf{r}_3$.

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x = 3 \\ & y - z = 5 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 4\mathbf{r}_2 & 5z = -20 \end{array}$$

Now simplify $\mathbf{r}_3$.

$$\begin{array}{ll} & x \qquad\quad = 3 \\ & \quad y - z = 5 \\ \mathbf{r}_3 \to \frac{1}{5}\mathbf{r}_3 & \qquad\quad z = -4 \end{array}$$

Then use $\mathbf{r}_3$ to eliminate the z-term from $\mathbf{r}_2$.

$$\begin{array}{ll} & x = 3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_3 & y = 1 \\ & z = -4 \end{array}$$

We conclude that there is a unique solution: $x = 3$, $y = 1$, $z = -4$.

Solution to Exercise C5

We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 & x & + & 3y & - & z & = & 4 \\ \mathbf{r}_2 & -x & + & 2y & - & 4z & = & 6 \\ \mathbf{r}_3 & x & + & 2y & & & = & 2 \end{array}$$

$$\begin{array}{lrcrcrcl} & x & + & 3y & - & z & = & 4 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 & & & 5y & - & 5z & = & 10 \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & & & -y & + & z & = & -2 \end{array}$$

$$\begin{array}{lrcrcrcl} & x & + & 3y & - & z & = & 4 \\ \mathbf{r}_2 \to \frac{1}{5}\mathbf{r}_2 & & & y & - & z & = & 2 \\ & & & -y & + & z & = & -2 \end{array}$$

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_2 & x & & & + & 2z & = & -2 \\ & & & y & - & z & = & 2 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 & & & & & 0 & = & 0 \end{array}$$

(The $\mathbf{r}_3$ equation $(0 = 0)$ gives no constraints on x, y and z.)

There are insufficient constraints on the unknowns to determine them uniquely, so the system has an infinite solution set.

(As both remaining equations involve a z-term, set z equal to the real parameter k.)

We write the general solution as

$$x = -2 - 2k, \quad y = 2 + k, \quad z = k, \quad k \in \mathbb{R}.$$

You may have spotted that $\mathbf{r}_2$ $(y - z = 2)$ and $\mathbf{r}_3$ $(-y + z = -2)$ were multiples of each other, and concluded earlier that there are infinitely many solutions; however, the solutions are still needed.

Solution to Exercise C6

We label the equations, and apply elementary operations to simplify the system.

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 & x & + & y & + & z & = & 6 \\ \mathbf{r}_2 & -x & + & y & - & 3z & = & -2 \\ \mathbf{r}_3 & 2x & + & y & + & 3z & = & 6 \end{array}$$

$$\begin{array}{lrcrcrcl} & x & + & y & + & z & = & 6 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 & & & 2y & - & 2z & = & 4 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 & & & -y & + & z & = & -6 \end{array}$$

$$\begin{array}{lrcrcrcl} & x & + & y & + & z & = & 6 \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 & & & y & - & z & = & 2 \\ & & & -y & + & z & = & -6 \end{array}$$

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & x & & & + & 2z & = & 4 \\ & & & y & - & z & = & 2 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_2 & & & & & 0 & = & -4 \end{array}$$

The $\mathbf{r}_3$ equation is $0 = -4$, so we conclude that the solution set is empty: the system is inconsistent.

You may have spotted that the system is inconsistent at an earlier stage and therefore stopped then.

Solution to Exercise C7

Let the equation of the plane be

$$ax + by + cz = d,$$

where a, b, c and d are real, and a, b and c are not all zero.

Substituting the points into the equation gives a system of three linear equations in the unknowns a, b and c. We label the equations and apply elementary operations to simplify the system.

$$\begin{array}{lrcrcrcl} \mathbf{r}_1 & a & & & + & 2c & = & d \\ \mathbf{r}_2 & & & 3b & + & 4c & = & d \\ \mathbf{r}_3 & a & + & b & + & 3c & = & d \end{array}$$

$$\begin{array}{lrcrcrcl} & a & & & + & 2c & = & d \\ & & & 3b & + & 4c & = & d \\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_1 & & & b & + & c & = & 0 \end{array}$$

$$\begin{array}{lrcrcrcl} & a & & & + & 2c & = & d \\ \mathbf{r}_2 \to \mathbf{r}_2 - 3\mathbf{r}_3 & & & & & c & = & d \\ & & & b & + & c & = & 0 \end{array}$$

$$\begin{array}{ll} & a \quad + 2c = d \\ \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 & \quad b + \ \ c = 0 \\ & \qquad\quad c = d \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_3 & a = -d \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_3 & b = -d \\ & c = d \end{array}$$

We conclude that this system has a unique solution (in terms of d): $a = -d,\ b = -d,\ c = d$.

We substitute these expressions into the equation for the plane to get

$$-dx - dy + dz = d.$$

Dividing through by $-d$ yields a simpler equation for the plane:

$$x + y - z = -1.$$

Solution to Exercise C8

The two unknowns are *my sister's age* and *my brother's age*; let us denote these by s and b (in years), respectively.

The first statement of the problem now translates to the equation

$$s + b = 40,$$

and the second statement to

$$b = s + 12.$$

We write these two equations in the usual form and label them.

$$\begin{array}{lr} \mathbf{r}_1 & s + b = 40 \\ \mathbf{r}_2 & -s + b = 12 \end{array}$$

We apply elementary operations to simplify this system.

$$\begin{array}{ll} & s + \ \ b = 40 \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 & \quad\ \ 2b = 52 \end{array}$$

$$\begin{array}{ll} & s + b = 40 \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2 & \quad\ \ b = 26 \end{array}$$

$$\begin{array}{ll} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 & s = 14 \\ & b = 26 \end{array}$$

The system has a unique solution: $s = 14$, $b = 26$.

The answer to the problem is that my sister is 14 years old.

Solution to Exercise C9

(a) The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 4 & -2 & 0 & -7 \\ 0 & 1 & 3 & 0 \\ 0 & -3 & 1 & 3 \end{array}\right).$$

(b) The corresponding system is

$$\begin{aligned} 2x + 3y \qquad\quad + 7w &= 1 \\ y - 7z \qquad\quad &= -1 \\ x \qquad + 3z - \ \ w &= 2. \end{aligned}$$

Solution to Exercise C10

(a) This matrix is not row-reduced as it does not have property 3.

(b) This matrix is row-reduced.

(c) This matrix is not row-reduced as it does not have property 4.

Solution to Exercise C11

(a) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \qquad &= \tfrac{1}{3} \\ x_2 &= \tfrac{2}{3}. \end{aligned}$$

The solution is $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$.

(b) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 \qquad + 6x_3 &= 0 \\ x_2 + 7x_3 &= 0 \\ 0 &= 1. \end{aligned}$$

The third equation cannot be satisfied, so there are no solutions.

(c) The augmented matrix corresponds to the system

$$\begin{aligned} x_1 + 3x_2 \qquad + 2x_4 \qquad &= -7, \\ x_3 - 3x_4 \qquad &= 8, \\ x_5 &= 11, \end{aligned}$$

that is,

$$x_1 = -7 - 3x_2 - 2x_4$$

$$x_3 = 8 + 3x_4$$

$$x_5 = 11.$$

Setting $x_2 = k$ and $x_4 = l$ ($k, l \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned}
x_1 &= -7 - 3k - 2l,\\
x_2 &= k,\\
x_3 &= 8 + 3l,\\
x_4 &= l,\\
x_5 &= 11.
\end{aligned}$$

(d) The augmented matrix corresponds to the system

$$\begin{array}{rcrcrcl}
x_1 & & & + & x_4 & = & 0\\
& x_2 & & + & 4x_4 & = & 3\\
& & x_3 & & & = & 0
\end{array}$$

that is,

$$\begin{aligned}
x_1 &= -x_4,\\
x_2 &= 3 - 4x_4,\\
x_3 &= 0.
\end{aligned}$$

Setting $x_4 = k$ ($k \in \mathbb{R}$), we obtain the general solution

$$\begin{aligned}
x_1 &= -k,\\
x_2 &= 3 - 4k,\\
x_3 &= 0,\\
x_4 &= k.
\end{aligned}$$

Solution to Exercise C12

(a) Strategy C1 gives the following sequence of row operations.

$$\begin{array}{l} \mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\\ \mathbf{r}_4 \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 1 & 4 & 5 & -1\\ 1 & 5 & 3 & 12 & 11 & 3\\ 3 & 15 & -1 & -4 & 3 & -6\\ -2 & -10 & 1 & 2 & -7 & 6 \end{array}\right)
\begin{array}{r} 15\\ 35\\ 10\\ -10 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1\\ \mathbf{r}_3 \to \mathbf{r}_3 - 3\mathbf{r}_1\\ \mathbf{r}_4 \to \mathbf{r}_4 + 2\mathbf{r}_1 \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 1 & 4 & 5 & -1\\ 0 & 0 & 2 & 8 & 6 & 4\\ 0 & 0 & -4 & -16 & -12 & -3\\ 0 & 0 & 3 & 10 & 3 & 4 \end{array}\right)
\begin{array}{r} 15\\ 20\\ -35\\ 20 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{2}\mathbf{r}_2\\ \\ \\ \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 1 & 4 & 5 & -1\\ 0 & 0 & 1 & 4 & 3 & 2\\ 0 & 0 & -4 & -16 & -12 & -3\\ 0 & 0 & 3 & 10 & 3 & 4 \end{array}\right)
\begin{array}{r} 15\\ 10\\ -35\\ 20 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2\\ \\ \mathbf{r}_3 \to \mathbf{r}_3 + 4\mathbf{r}_2\\ \mathbf{r}_4 \to \mathbf{r}_4 - 3\mathbf{r}_2 \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 0 & 0 & 2 & -3\\ 0 & 0 & 1 & 4 & 3 & 2\\ 0 & 0 & 0 & 0 & 0 & 5\\ 0 & 0 & 0 & -2 & -6 & -2 \end{array}\right)
\begin{array}{r} 5\\ 10\\ 5\\ -10 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \leftrightarrow \mathbf{r}_4\\ \\ \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 0 & 0 & 2 & -3\\ 0 & 0 & 1 & 4 & 3 & 2\\ 0 & 0 & 0 & -2 & -6 & -2\\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right)
\begin{array}{r} 5\\ 10\\ -10\\ 5 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\frac{1}{2}\mathbf{r}_3\\ \\ \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 0 & 0 & 2 & -3\\ 0 & 0 & 1 & 4 & 3 & 2\\ 0 & 0 & 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right)
\begin{array}{r} 5\\ 10\\ 5\\ 5 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_3\\ \\ \\ \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 0 & 0 & 2 & -3\\ 0 & 0 & 1 & 0 & -9 & -2\\ 0 & 0 & 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right)
\begin{array}{r} 5\\ -10\\ 5\\ 5 \end{array}$$

$$\begin{array}{l} \\ \\ \\ \mathbf{r}_4 \to \frac{1}{5}\mathbf{r}_4 \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 0 & 0 & 2 & -3\\ 0 & 0 & 1 & 0 & -9 & -2\\ 0 & 0 & 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\begin{array}{r} 5\\ -10\\ 5\\ 1 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 3\mathbf{r}_4\\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_4\\ \mathbf{r}_3 \to \mathbf{r}_3 - \mathbf{r}_4\\ \\ \end{array}
\left(\begin{array}{rrrrrr} 1 & 5 & 0 & 0 & 2 & 0\\ 0 & 0 & 1 & 0 & -9 & 0\\ 0 & 0 & 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\begin{array}{r} 8\\ -8\\ 4\\ 1 \end{array}$$

This is the row-reduced form of the matrix.

(b) Strategy C1 gives the following sequence of row operations.

Your sequence may differ from this because in the first step shown below (which corresponds to step 2 of the strategy) row 1 and row 5 are interchanged, whereas row 1 could have been interchanged with another row. However, your final row-reduced matrix should be the same as this one, since (as you will see in Theorem C1) the row-reduced form of a matrix is unique.

$$\begin{array}{l} \mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3\\ \mathbf{r}_4\\ \mathbf{r}_5 \end{array}
\left(\begin{array}{rrrr} 0 & -8 & 8 & -14\\ -1 & 0 & -4 & -6\\ -1 & 8 & -12 & 8\\ 2 & 8 & 0 & 24\\ 1 & 4 & 0 & 14 \end{array}\right)
\begin{array}{r} -14\\ -11\\ 3\\ 34\\ 19 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \leftrightarrow \mathbf{r}_5\\ \\ \\ \\ \\ \end{array}
\left(\begin{array}{rrrr} 1 & 4 & 0 & 14\\ -1 & 0 & -4 & -6\\ -1 & 8 & -12 & 8\\ 2 & 8 & 0 & 24\\ 0 & -8 & 8 & -14 \end{array}\right)
\begin{array}{r} 19\\ -11\\ 3\\ 34\\ -14 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 + \mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 + \mathbf{r}_1 \\ \mathbf{r}_4 \to \mathbf{r}_4 - 2\mathbf{r}_1 \\ \\ \end{array} \begin{pmatrix} 1 & 4 & 0 & 14 \\ 0 & 4 & -4 & 8 \\ 0 & 12 & -12 & 22 \\ 0 & 0 & 0 & -4 \\ 0 & -8 & 8 & -14 \end{pmatrix} \begin{array}{r} 19 \\ 8 \\ 22 \\ -4 \\ -14 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{4}\mathbf{r}_2 \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 4 & 0 & 14 \\ 0 & 1 & -1 & 2 \\ 0 & 12 & -12 & 22 \\ 0 & 0 & 0 & -4 \\ 0 & -8 & 8 & -14 \end{pmatrix} \begin{array}{r} 19 \\ 2 \\ 22 \\ -4 \\ -14 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 4\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 12\mathbf{r}_2 \\ \\ \mathbf{r}_5 \to \mathbf{r}_5 + 8\mathbf{r}_2 \end{array} \begin{pmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{array}{r} 11 \\ 2 \\ -2 \\ -4 \\ 2 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to -\frac{1}{2}\mathbf{r}_3 \\ \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 4 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{array}{r} 11 \\ 2 \\ 1 \\ -4 \\ 2 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 6\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_3 \\ \\ \mathbf{r}_4 \to \mathbf{r}_4 + 4\mathbf{r}_3 \\ \mathbf{r}_5 \to \mathbf{r}_5 - 2\mathbf{r}_3 \end{array} \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{r} 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}$$

This is the row-reduced form of the matrix.

Solution to Exercise C13

We have

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1 \\ \\ \end{array} \begin{pmatrix} 1 & 3 & 1 & 2 \\ -1 & 1 & 4 & 5 \\ 0 & 3 & 4 & 9 \end{pmatrix} \begin{array}{r} 7 \\ 9 \\ 16 \end{array}$$

The row operation has created a 1 in the correct position in the current row, but it is not a leading 1 because it has changed the 0 at the beginning of the row to -1. Performing this row operation has destroyed the progress made so far on the matrix: the first column no longer contains a leading 1 with only zeros above and below.

Solution to Exercise C14

We follow Strategy C2 and row-reduce the augmented matrix.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \left(\begin{array}{ccccc|c} 1 & -4 & -4 & 3 & 6 & 2 \\ 2 & -5 & -6 & 6 & 9 & 3 \\ 2 & 4 & 0 & 9 & 2 & 0 \end{array}\right) \begin{array}{r} 4 \\ 9 \\ 17 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left(\begin{array}{ccccc|c} 1 & -4 & -4 & 3 & 6 & 2 \\ 0 & 3 & 2 & 0 & -3 & -1 \\ 0 & 12 & 8 & 3 & -10 & -4 \end{array}\right) \begin{array}{r} 4 \\ 1 \\ 9 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2 \\ \\ \end{array} \left(\begin{array}{ccccc|c} 1 & -4 & -4 & 3 & 6 & 2 \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 12 & 8 & 3 & -10 & -4 \end{array}\right) \begin{array}{r} 4 \\ \frac{1}{3} \\ 9 \end{array}$$

(Note that here we cannot find a row operation that could be performed instead of $\mathbf{r}_2 \to \frac{1}{3}\mathbf{r}_2$ to create a leading 1 while avoiding fractions.)

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 4\mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 12\mathbf{r}_2 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -\frac{4}{3} & 3 & 2 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 3 & 2 & 0 \end{array}\right) \begin{array}{r} \frac{16}{3} \\ \frac{1}{3} \\ 5 \end{array}$$

$$\begin{array}{l} \\ \\ \mathbf{r}_3 \to \frac{1}{3}\mathbf{r}_3 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -\frac{4}{3} & 3 & 2 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right) \begin{array}{r} \frac{16}{3} \\ \frac{1}{3} \\ \frac{5}{3} \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 3\mathbf{r}_3 \\ \\ \\ \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -\frac{4}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right) \begin{array}{r} \frac{1}{3} \\ \frac{1}{3} \\ \frac{5}{3} \end{array}$$

This matrix is in row-reduced form.

The corresponding system is

$$\begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 \qquad\qquad\quad &= \tfrac{2}{3} \\ x_2 + \tfrac{2}{3}x_3 \qquad - \quad x_5 &= -\tfrac{1}{3} \\ x_4 + \tfrac{2}{3}x_5 &= 0, \end{aligned}$$

that is,

$$\begin{aligned} x_1 &= \tfrac{2}{3} + \tfrac{4}{3}x_3 \\ x_2 &= -\tfrac{1}{3} - \tfrac{2}{3}x_3 + x_5 \\ x_4 &= -\tfrac{2}{3}x_5. \end{aligned}$$

Setting $x_3 = k$ and $x_5 = l$ $(k, l \in \mathbb{R})$, we obtain the general solution

$$\begin{aligned} x_1 &= \tfrac{2}{3} + \tfrac{4}{3}k, \\ x_2 &= -\tfrac{1}{3} - \tfrac{2}{3}k + l, \\ x_3 &= k, \\ x_4 &= -\tfrac{2}{3}l, \\ x_5 &= l \quad (k, l \in \mathbb{R}). \end{aligned}$$

Solution to Exercise C15

(a) $\begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & 55 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -2 & 54 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 2 & 55 \end{pmatrix}$

(c) This sum is undefined since the matrices are of different sizes.

(d) $$\begin{pmatrix} 0 & 6 & -2 \\ 1 & 8 & 2 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 9 \\ 1 & 0 & 4 \\ 3 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 7 \\ 2 & 8 & 6 \\ 3 & -1 & 5 \end{pmatrix}$$

Solution to Exercise C16

(a) This difference is undefined since the matrices are of different sizes.

(b) $$\begin{pmatrix} 5 & 8 & 12 \\ 7 & 2 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 10 & 2 \\ 4 & 9 & 21 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 10 \\ 3 & -7 & -22 \end{pmatrix}$$

Solution to Exercise C17

(a) $4\mathbf{A} = 4\begin{pmatrix} 5 & -3 \\ 2 & 3 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -12 \\ 8 & 12 \\ -4 & 0 \end{pmatrix}$

(b) $4\mathbf{B} = 4\begin{pmatrix} 2 & 1 \\ -2 & -7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ -8 & -28 \\ 12 & 20 \end{pmatrix}$

(c) $$4\mathbf{A} + 4\mathbf{B} = \begin{pmatrix} 20 & -12 \\ 8 & 12 \\ -4 & 0 \end{pmatrix} + \begin{pmatrix} 8 & 4 \\ -8 & -28 \\ 12 & 20 \end{pmatrix} = \begin{pmatrix} 28 & -8 \\ 0 & -16 \\ 8 & 20 \end{pmatrix}$$

(d) $$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 5 & -3 \\ 2 & 3 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ -2 & -7 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 0 & -4 \\ 2 & 5 \end{pmatrix},$$

thus

$$4(\mathbf{A} + \mathbf{B}) = 4\begin{pmatrix} 7 & -2 \\ 0 & -4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 28 & -8 \\ 0 & -16 \\ 8 & 20 \end{pmatrix}$$

(Note that $4(\mathbf{A} + \mathbf{B}) = 4\mathbf{A} + 4\mathbf{B}$.)

Solution to Exercise C18

(a) We add corresponding entries of the three matrices $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ and $\mathbf{C} = (c_{ij})$. The (i, j)-entry of the matrix $\mathbf{A} + (\mathbf{B} + \mathbf{C})$ is $a_{ij} + (b_{ij} + c_{ij})$, and that of $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ is $(a_{ij} + b_{ij}) + c_{ij}$. Now, a_{ij}, b_{ij} and c_{ij} are real numbers, so $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$. Therefore

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}.$$

(b) We add corresponding entries of the two matrices. The (i, j)-entry of the matrix $\mathbf{A} + \mathbf{0}$ is $a_{ij} + 0 = a_{ij}$. Therefore $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

Matrix addition is commutative (property A5), so $\mathbf{0} + \mathbf{A} = \mathbf{A}$ also.

(c) Let $\mathbf{A} = (a_{ij})$, so $-\mathbf{A} = (-a_{ij})$. We add corresponding entries: the (i, j)-entry of the matrix $\mathbf{A} + (-\mathbf{A})$ is $a_{ij} + (-a_{ij}) = 0$. Thus the matrix $\mathbf{A} + (-\mathbf{A})$ is the zero matrix $\mathbf{0}$.

Matrix addition is commutative (property A5), so $-\mathbf{A} + \mathbf{A} = \mathbf{A} + (-\mathbf{A})$. Thus $-\mathbf{A} + \mathbf{A}$ is also the zero matrix $\mathbf{0}$.

Solution to Exercise C19

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. Then the (i, j)-entry of $k(\mathbf{A} + \mathbf{B})$ is $k(a_{ij} + b_{ij})$.

Now, $k\mathbf{A} = (ka_{ij})$ and $k\mathbf{B} = (kb_{ij})$, so the (i, j)-entry of $k\mathbf{A} + k\mathbf{B}$ is $ka_{ij} + kb_{ij} = k(a_{ij} + b_{ij})$ since $a_{i,j}$, $b_{i,j}$ and k are real numbers.

The (i, j)-entries of $k(\mathbf{A} + \mathbf{B})$ and $k\mathbf{A} + k\mathbf{B}$ are equal. Thus

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}.$$

Solution to Exercise C20

(a) The product of a 3×2 matrix with a 2×1 matrix is a 3×1 matrix:

$$\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}.$$

(b) The product of a 1×2 matrix with a 2×2 matrix is a 1×2 matrix:

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 14 \end{pmatrix}.$$

(c) This product is not defined, since the first matrix has 1 column and the second has 2 rows.

(d) The product of a 2×1 matrix with a 1×3 matrix is a 2×3 matrix:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -4 \\ 6 & 0 & -8 \end{pmatrix}.$$

(e) The product of a 2×3 matrix with a 3×3 matrix is a 2×3 matrix:

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 1 & 3 & 0 \\ 4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 1 \\ 9 & 16 & -1 \end{pmatrix}.$$

Solution to Exercise C21

(a) We first prove the *if* statement.

Suppose **A** and **B** are square matrices of the same size, then the product **AB** can be formed because **A** has the same number of columns as **B** has rows. Likewise, the product **BA** can be formed. Both the products **AB** and **BA** will be square matrices the same size as **A** and **B**.

We now prove the *only if* statement.

Suppose the products **AB** and **BA** are the same size, and suppose **A** is an $m \times n$ matrix and **B** is a $p \times r$ matrix.

Since the product **AB** is defined, then we must have $n = p$, and therefore the product **AB** is an $m \times r$ matrix.

Since the product **BA** is defined, then we must have $r = m$, and therefore the product **BA** is a $p \times n$ matrix.

Since the products **AB** and **BA** are the same size, $m = p$ and $r = n$, but this combined with $n = p$ and $r = m$, implies that $p = r = m = n$. Therefore, both **A** and **B** are square matrices of the same size.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $\mathbf{AB} \neq \mathbf{BA}$ in this case. It follows that matrix multiplication is not commutative even for square matrices of the same size.

(There are infinitely many possible examples here; however, the trick when looking for a counterexample is to do as little work as possible: setting several of the entries to zero makes the multiplication easier!)

Solution to Exercise C22

The product of a 2×2 matrix with a 2×2 matrix is a 2×2 matrix.

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 28 \end{pmatrix}$

(b) $\mathbf{BA} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 28 \end{pmatrix}$

Note that **AB** and **BA** are equal in this case.

(c) Matrix multiplication is associative, so

$$\begin{aligned} \mathbf{ABC} &= (\mathbf{AB})\mathbf{C} \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} \right) \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 0 \\ 0 & 28 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 0 \\ 0 & 336 \end{pmatrix}. \end{aligned}$$

(If you worked out $= \mathbf{A}(\mathbf{BC})$ then you would have got the same final answer here.)

Solution to Exercise C23

(a) The matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is upper triangular.

(b) The matrix $\begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$ is diagonal (so it is also both upper and lower triangular).

(c) The matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ is *not* diagonal, upper triangular or lower triangular.

(d) The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is lower triangular.

Solution to Exercise C24

The (i, j)-entry of the product $\mathbf{I}_m\mathbf{A}$ is obtained by multiplying each entry in the ith row of $\mathbf{I}_m$ by the corresponding entry in the jth column of $\mathbf{A}$ and adding the results. Now, the ith row of $\mathbf{I}_m$ has a 1 in the ith position and zeros elsewhere. Therefore the only non-zero term in this sum is the ith entry of the jth column of $\mathbf{A}$, that is, the (i, j)-entry of $\mathbf{A}$. Thus $\mathbf{I}_m\mathbf{A} = \mathbf{A}$.

The (i, j)-entry of the product $\mathbf{A}\mathbf{I}_n$ is obtained by multiplying each entry in the ith row of $\mathbf{A}$ by the corresponding entry in the jth column of $\mathbf{I}_n$ and adding the results. Now, the jth column of $\mathbf{I}_n$ has a 1 in the jth position and zeros elsewhere. Therefore the only non-zero term in this sum is the jth entry of the ith row of $\mathbf{A}$, that is, the (i, j)-entry of $\mathbf{A}$. Thus $\mathbf{A}\mathbf{I}_n = \mathbf{A}$.

Solution to Exercise C25

(a) The transpose of a 3×2 matrix is a 2×3 matrix:

$$\begin{pmatrix} 1 & 4 \\ 0 & 2 \\ -6 & 10 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -6 \\ 4 & 2 & 10 \end{pmatrix}.$$

(b) The transpose of a 3×3 matrix is a 3×3 matrix:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & -5 \\ 4 & 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 7 \\ 2 & -5 & 0 \end{pmatrix}.$$

(c) The transpose of a 1×3 matrix is a 3×1 matrix:

$$\begin{pmatrix} 10 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 10 \\ 4 \\ 6 \end{pmatrix}.$$

(d) The transpose of a 2×2 matrix is a 2×2 matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solution to Exercise C26

(a) Here,

$$\mathbf{A}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix},$$

$$\mathbf{B}^T = \begin{pmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{pmatrix}$$

and

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 18 \end{pmatrix}.$$

So

$$(\mathbf{A} + \mathbf{B})^T = \begin{pmatrix} 8 & 12 & 16 \\ 10 & 14 & 18 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{A}^T + \mathbf{B}^T &= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 9 & 11 \\ 8 & 10 & 12 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 12 & 16 \\ 10 & 14 & 18 \end{pmatrix} \\ &= (\mathbf{A} + \mathbf{B})^T. \end{aligned}$$

(b) Here,

$$\mathbf{C}^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{AC} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 4 \\ 11 & 6 \end{pmatrix}.$$

So

$$(\mathbf{AC})^T = \begin{pmatrix} 3 & 7 & 11 \\ 2 & 4 & 6 \end{pmatrix}.$$

The product $\mathbf{A}^T\mathbf{C}^T$ cannot be formed, since $\mathbf{A}^T$ is a 2×3 matrix and $\mathbf{C}^T$ is a 2×2 matrix.
The product $\mathbf{C}^T\mathbf{A}^T$ does, however, exist:

$$\begin{aligned} \mathbf{C}^T\mathbf{A}^T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 7 & 11 \\ 2 & 4 & 6 \end{pmatrix} \\ &= (\mathbf{AC})^T. \end{aligned}$$

Solution to Exercise C27

Suppose, for a contradiction, that there exists a matrix $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathbf{AB} = \mathbf{I}$, that is,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiplying the matrices on the left-hand side gives:

$$\begin{pmatrix} a-c & b-d \\ -a+c & -b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Looking at the entries in the first column, we must have $a-c=1$ and $-a+c=0$, that is, $a-c=1$ and $a-c=0$. This contradiction shows that there exists no such matrix $\mathbf{B}$. (The same conclusion arises from looking at the entries in the second column.)

Solution to Exercise C28

The equation $\mathbf{II} = \mathbf{I}$ shows that $\mathbf{I}$ is invertible, with inverse $\mathbf{I}$.

Solution to Exercise C29

To prove that $\mathbf{AB}$ is invertible, with inverse $\mathbf{B}^{-1}\mathbf{A}^{-1}$, we have to show that

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I} = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}).$$

By the associative property (M2),

$$\begin{aligned} (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} \\ &= \mathbf{AIA}^{-1} \\ &= \mathbf{AA}^{-1} = \mathbf{I}, \end{aligned}$$

and, similarly,

$$\begin{aligned} (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) &= \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} \\ &= \mathbf{B}^{-1}\mathbf{IB} \\ &= \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}. \end{aligned}$$

Therefore $\mathbf{AB}$ is invertible, with inverse $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Solution to Exercise C30

(a) We row-reduce $(\mathbf{A} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \end{array} \quad \left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} 7 \\ 6 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1 \\ \\ \end{array} \quad \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 4 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} \frac{7}{2} \\ 6 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 4\mathbf{r}_1 \end{array} \quad \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -7 & -2 & 1 \end{array}\right) \begin{array}{c} \frac{7}{2} \\ -8 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\frac{1}{7}\mathbf{r}_2 \end{array} \quad \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} \end{array}\right) \begin{array}{c} \frac{7}{2} \\ \frac{8}{7} \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{14} & \frac{2}{7} \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} \end{array}\right) \begin{array}{c} \frac{17}{14} \\ \frac{8}{7} \end{array}$$

The left half has been reduced to $\mathbf{I}$, so $\mathbf{A}$ is invertible; its inverse is

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{14} & \frac{2}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{pmatrix}.$$

(b) We row-reduce $(\mathbf{B} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 1 & 0 & 0 \\ 2 & 1 & -6 & 0 & 1 & 0 \\ -3 & -1 & 9 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ -2 \\ 6 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - 2\mathbf{r}_1 \\ \mathbf{r}_3 \to \mathbf{r}_3 + 3\mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 3 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to -\mathbf{r}_2 \\ \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 - \mathbf{r}_2 \\ \\ \mathbf{r}_3 \to \mathbf{r}_3 - 2\mathbf{r}_2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right) \begin{array}{c} -1 \\ 0 \\ 3 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \mathbf{r}_1 + 2\mathbf{r}_3 \\ \mathbf{r}_2 \to \mathbf{r}_2 + 2\mathbf{r}_3 \\ \\ \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5 & 2 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right) \begin{array}{c} 5 \\ 6 \\ 3 \end{array}$$

The left half has been reduced to $\mathbf{I}$, so $\mathbf{B}$ is invertible; its inverse is

$$\mathbf{B}^{-1} = \begin{pmatrix} -3 & 5 & 2 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

(c) We row-reduce $(\mathbf{C} \mid \mathbf{I})$.

$$\begin{array}{l} \mathbf{r}_1\\ \mathbf{r}_2\\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{rrr|rrr} 2 & 4 & 6 & 1 & 0 & 0\\ 1 & 2 & 4 & 0 & 1 & 0\\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} 13\\ 8\\ 21 \end{array}$$

$$\begin{array}{l} \mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1\\ \\ \\ \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 2 & 3 & \frac{1}{2} & 0 & 0\\ 1 & 2 & 4 & 0 & 1 & 0\\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{r} \frac{13}{2}\\ 8\\ 21 \end{array}$$

$$\begin{array}{l} \\ \mathbf{r}_2 \to \mathbf{r}_2 - \mathbf{r}_1\\ \mathbf{r}_3 \to \mathbf{r}_3 - 5\mathbf{r}_1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 2 & 3 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & -\frac{1}{2} & 1 & 0\\ 0 & 0 & -10 & -\frac{5}{2} & 0 & 1 \end{array}\right) \begin{array}{r} \frac{13}{2}\\ \frac{3}{2}\\ -\frac{23}{2} \end{array}$$

The usual strategy for row-reduction has created a leading 1 in the second row that does not lie on the main diagonal of the left half. Hence the left half cannot reduce to $\mathbf{I}$, and therefore $\mathbf{C}$ is not invertible.

Solution to Exercise C31

The matrix form of the system is

$$\begin{pmatrix} 1 & 1 & 2\\ -1 & 0 & -4\\ 3 & 2 & 10 \end{pmatrix} \begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 1\\ 2\\ -1 \end{pmatrix}.$$

Multiplying this equation on the left by the inverse of the coefficient matrix gives the solution

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & -2\\ -1 & 2 & 1\\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1\\ 2\\ -1 \end{pmatrix} = \begin{pmatrix} 0\\ 2\\ -\frac{1}{2} \end{pmatrix};$$

that is, $x = 0$, $y = 2$, $z = -\frac{1}{2}$.

Solution to Exercise C32

(a)

$$\underset{\substack{\text{elementary}\\ \text{matrix}\\ \text{associated with}\\ \mathbf{r}_1 \to 5\mathbf{r}_1}}{\begin{pmatrix} 5 & 0\\ 0 & 1 \end{pmatrix}} \underset{\mathbf{A}}{\begin{pmatrix} 1 & 2 & 3\\ 3 & 2 & 1 \end{pmatrix}} = \underset{\substack{\text{matrix}\\ \text{obtained when}\\ \mathbf{r}_1 \to 5\mathbf{r}_1\\ \text{is performed on } \mathbf{A}}}{\begin{pmatrix} 5 & 10 & 15\\ 3 & 2 & 1 \end{pmatrix}}$$

(b)

$$\underset{\substack{\text{elementary}\\ \text{matrix}\\ \text{associated with}\\ \mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4}}{\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}} \underset{\mathbf{B}}{\begin{pmatrix} 1 & 2\\ 3 & 4\\ 5 & 6\\ 7 & 8 \end{pmatrix}} = \underset{\substack{\text{matrix}\\ \text{obtained when}\\ \mathbf{r}_2 \to \mathbf{r}_2 + 3\mathbf{r}_4\\ \text{is performed on } \mathbf{B}}}{\begin{pmatrix} 1 & 2\\ 24 & 28\\ 5 & 6\\ 7 & 8 \end{pmatrix}}$$

Solution to Exercise C33

(a) The inverse elementary row operation of $\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2$ is $\mathbf{r}_1 \to \mathbf{r}_1 + 2\mathbf{r}_2$.

The working below shows the sequence of two row operations performed on $\mathbf{A}$.

$$\begin{array}{ll} \begin{array}{l} \mathbf{r}_1\\ \mathbf{r}_2 \end{array} & \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{pmatrix}\\[2ex] \mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_2 & \begin{pmatrix} -7 & -8 & -9\\ 4 & 5 & 6 \end{pmatrix}\\[2ex] \mathbf{r}_1 \to \mathbf{r}_1 + 2\mathbf{r}_2 & \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{pmatrix} \end{array}$$

(b) The inverse elementary row operation of $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$ is $\mathbf{r}_1 \leftrightarrow \mathbf{r}_2$.

The working below shows the sequence of two row operations performed on $\mathbf{A}$.

$$\begin{array}{ll} \begin{array}{l} \mathbf{r}_1\\ \mathbf{r}_2 \end{array} & \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{pmatrix}\\[2ex] \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 & \begin{pmatrix} 4 & 5 & 6\\ 1 & 2 & 3 \end{pmatrix}\\[2ex] \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 & \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{pmatrix} \end{array}$$

(c) The inverse elementary row operation of $\mathbf{r}_2 \to -3\mathbf{r}_2$ is $\mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2$.

The working below shows the sequence of two row operations performed on $\mathbf{A}$.

$$\begin{array}{ll} \begin{array}{l} \mathbf{r}_1\\ \mathbf{r}_2 \end{array} & \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{pmatrix}\\[2ex] \mathbf{r}_2 \to -3\mathbf{r}_2 & \begin{pmatrix} 1 & 2 & 3\\ -12 & -15 & -18 \end{pmatrix}\\[2ex] \mathbf{r}_2 \to -\frac{1}{3}\mathbf{r}_2 & \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6 \end{pmatrix} \end{array}$$

Solution to Exercise C34

Matrix $\mathbf{A}$ has associated elementary row operation $\mathbf{r}_1 \to 2\mathbf{r}_1$, which has inverse $\mathbf{r}_1 \to \frac{1}{2}\mathbf{r}_1$. The inverse of $\mathbf{A}$ is the elementary matrix associated with this inverse row operation, which is

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

Solution to Exercise C35

(a) $\begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = (5 \times 2) - (1 \times 4) = 6$

(b) $\begin{vmatrix} 10 & -4 \\ -5 & 2 \end{vmatrix} = (10 \times 2) - (-4 \times (-5)) = 0$

(c) $\begin{vmatrix} 7 & 3 \\ 17 & 2 \end{vmatrix} = (7 \times 2) - (3 \times 17) = -37$

Solution to Exercise C36

(a) First we evaluate the determinant of the matrix:

$$\begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = (4 \times 6) - (2 \times 5) = 14.$$

This determinant is non-zero, so the matrix is invertible. We use the formula to find the inverse:

$$\begin{pmatrix} 4 & 2 \\ 5 & 6 \end{pmatrix}^{-1} = \frac{1}{14}\begin{pmatrix} 6 & -2 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ -\frac{5}{14} & \frac{2}{7} \end{pmatrix}.$$

(b) First we evaluate the determinant of the matrix:

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = (1 \times 1) - (1 \times (-1)) = 2.$$

This determinant is non-zero, so the matrix is invertible. We use the formula to find the inverse:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(c) First we evaluate the determinant of the matrix:

$$\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = (1 \times 1) - (-1 \times (-1)) = 0.$$

This determinant is 0, so the matrix is not invertible.

Solution to Exercise C37

(a) We have

$$\begin{aligned}
&\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & -1 \\ 0 & -1 & 1 \end{vmatrix} \\
&\quad = 3\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & -1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 0 \\ 0 & -1 \end{vmatrix} \\
&\quad = 3((0 \times 1) - (-1 \times (-1))) \\
&\qquad - 2((4 \times 1) - (-1 \times 0)) \\
&\qquad + ((4 \times (-1)) - (0 \times 0)) \\
&\quad = -15
\end{aligned}$$

(b) We have

$$\begin{aligned}
\begin{vmatrix} 2 & 10 & 0 \\ 3 & -1 & 2 \\ 5 & 9 & 2 \end{vmatrix} &= 2\begin{vmatrix} -1 & 2 \\ 9 & 2 \end{vmatrix} - 10\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} + 0 \\
&= 2((-1 \times 2) - (2 \times 9)) \\
&\quad - 10((3 \times 2) - (2 \times 5)) \\
&= 0
\end{aligned}$$

Solution to Exercise C38

The cofactor A_{13} is $(-1)^{1+3} = (-1)^4 = 1$ times the determinant of the submatrix obtained by removing the top row and third column of $\mathbf{A}$:

$$A_{13} = \begin{vmatrix} 2 & 3 & 5 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 5 & 2 & 3 \\ 5 & 1 & 3 & 4 \end{vmatrix}.$$

The cofactor A_{45} is $(-1)^{4+5} = (-1)^9 = -1$ times the determinant of the submatrix obtained by removing the fourth row and fifth column of $\mathbf{A}$:

$$A_{45} = -\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 \end{vmatrix}.$$

Solution to Exercise C39

We apply Strategy C5:

$$\begin{aligned}
&\begin{vmatrix} 0 & 2 & 1 & -1 \\ -3 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{vmatrix} \\
&= 0 - 2\begin{vmatrix} -3 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} + \begin{vmatrix} -3 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix} \\
&\qquad - (-1)\begin{vmatrix} -3 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 2 \end{vmatrix} \\
&= -2\left(-3\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}\right) \\
&\qquad + \left(-3\begin{vmatrix} 0 & 0 \\ 4 & 0 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}\right) \\
&\qquad + \left(-3\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 0 + 0\right) \\
&= (-2)(-2) + (-1)4 + (-3)(-4) \\
&= 12.
\end{aligned}$$

Solution to Exercise C40

Here,

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} -3 & 1 \\ 2 & -4 \end{vmatrix} \\
&= (-3 \times (-4)) - (1 \times 2) = 10, \\
\det \mathbf{B} &= \begin{vmatrix} 1 & 1 \\ -2 & 5 \end{vmatrix} \\
&= (1 \times 5) - (1 \times (-2)) = 7
\end{aligned}$$

and

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix},$$

so

$$\det(\mathbf{A} + \mathbf{B}) = (-2 \times 1) - (2 \times 0) = -2.$$

We have $\det \mathbf{A} + \det \mathbf{B} = 10 + 7 = 17$, and so $\det(\mathbf{A} + \mathbf{B})$ is not equal to $\det \mathbf{A} + \det \mathbf{B}$.

$$\mathbf{AB} = \begin{pmatrix} -5 & 2 \\ 10 & -18 \end{pmatrix},$$

so

$$\det(\mathbf{AB}) = (-5 \times (-18)) - (2 \times 10) = 70.$$

We have $(\det \mathbf{A})(\det \mathbf{B}) = 10 \times 7 = 70$, and so

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}).$$

Solution to Exercise C41

(a) We apply Strategy C5:

$$\begin{aligned}
\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} &= 0 - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 \\
&= -1.
\end{aligned}$$

(b) We apply Strategy C5:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} - 0 + 0 - 0 \\
&= \begin{vmatrix} k & 0 \\ 0 & 1 \end{vmatrix} - 0 + 0 \\
&= (k \times 1) - (0 \times 0) \\
&= k.
\end{aligned}$$

(c) We evaluate the determinant:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix} &= (1 \times 1) - (0 \times k) \\
&= 1.
\end{aligned}$$

Solution to Exercise C42

First notice that

$$-2\begin{pmatrix} 1 & -2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 4 & -8 \end{pmatrix},$$

that is, the first and third rows of $\mathbf{A}$ are proportional. Therefore, by Theorem C16,

$$\det \mathbf{A} = 0.$$

Solution to Exercise C43

We interchange the first and third rows, and apply Theorems C14 and C15, giving

$$\det \mathbf{A} = \begin{vmatrix} 10 & 3 & -4 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 0 & 6 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 10 & 3 & -4 & 2 \\ -1 & 2 & 1 & 0 \end{vmatrix}.$$

We use Strategy C5 to evaluate this determinant:

$$\begin{aligned} \det \mathbf{A} &= (-1) \left(0 - 6 \begin{vmatrix} 0 & 0 & 1 \\ 10 & -4 & 2 \\ -1 & 1 & 0 \end{vmatrix} + 0 - 0 \right) \\ &= 6 \left(0 - 0 + \begin{vmatrix} 10 & -4 \\ -1 & 1 \end{vmatrix} \right) \\ &= 6\left((10 \times 1) - (-4 \times (-1))\right) \\ &= 36. \end{aligned}$$